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FRACTIONAL MATHIEU EQUATION

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ABSTRACT

After reviewing the concept of fractional derivative, we derive expressions for the transition curves separating regions of stability from regions of instability in the ODE:

$$x'' + (\delta + \varepsilon \cos t)x + cD^{\alpha}x = 0$$

where $D^{\alpha}x$ is the order α derivative of $x(t)$, where $0 < \alpha < 1$. We use the method of harmonic balance and obtain both a lowest order approximation as well as a higher order approximation for the $n = 1$ transition curves. We also obtain an expression for the $n = 0$ transition curves.

1 INTRODUCTION

The fractional calculus and fractional differential equations have recently become increasingly important topics in the literature of engineering, science and applied mathematics. Application areas include viscoelasticity, electromagnetics, heat conduction, control theory and diffusion [17], [10], [9], [6], [8], [11]. One reason for the interest in this subject comes from applications which involve new ways of modeling physical systems using tools from fractional calculus. For example, consider the dynamics of a system which involves the motion of a rheological specimen which exhibits both elasticity and dissipation. Traditional models of such a system might be based on the following familiar linear differential equation:

$$x'' + cx' + kx = 0 \tag{1}$$

However, an alternative approach would be to combine the effects of stiffness and damping in a single term:

$$x'' + \mu D^{\alpha}x = 0 \tag{2}$$

where $D^{\alpha}x$ is the order α derivative of $x(t)$, where $0 < \alpha < 1$ is a parameter, and where μ is a coefficient of "fractance". As α varies from 0 to 1, the relative importance of the stiffness and damping terms may be adjusted. See e.g. [26]. Note that eq.(2) is linear.

Recent literature has dealt with the treatment of diverse fractional differential equations. These include:

1. fractional linear oscillator [27], [21], [29], [14].

$$x'' + x + \mu D^{\alpha}x = 0 \tag{3}$$

2. another fractional linear oscillator [10], [22], [29].

$$D^{\alpha}x + x = 0 \tag{4}$$

3. fractional Duffing equation [27], [4]

$$x'' + \varepsilon D^{\alpha} x + x^3 = 0 \tag{5}$$

4. fractional van der Pol equation [28]

$$x'' - \varepsilon(1 - x^2) D^{\alpha} x + x = 0 \tag{6}$$

5. another fractional van der Pol equation [2], [25]

$$D^{\alpha} x - \varepsilon(1 - x^2) x' + x = 0 \tag{7}$$

6. fractional jerk model [1]

$$D^{\alpha+2} x + \mu\, D^{\alpha+1} x + D^{\alpha} x = f(x) \tag{8}$$

7. fractional wave equation [10]

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u}{\partial x^2} \tag{9}$$

8. equations exhibiting chaos [16], [7], [5].

It is the purpose of the present work to extend the treatment of Mathieu's equation,

$$x'' + (\delta + \varepsilon \cos t) x = 0, \tag{10}$$

being an equation which is important in questions of stability of motion as well as in systems which are parametrically excited, to include the effect of a fractional derivative term:

$$x'' + (\delta + \varepsilon \cos t) x + c D^{\alpha} x = 0 \tag{11}$$

In the case that $\alpha = 1$, eq.(11) represents the familiar damped Mathieu equation [19].

We begin the paper with a brief introduction to the fractional calculus. See e.g. [15], [20], [17], [12], [23].

2 FRACTIONAL DERIVATIVES

We offer a formal derivation of the key formula which defines the fractional derivative of a function $x(t)$, beginning with an intuitive definition of the fractional derivative of t^k, $D^{\alpha} t^k$. By "formal" we mean that issues of convergence will be ignored. This formal derivation may thus be thought of as a plausibility argument rather than a rigorous derivation. After Ross [20], we note that

$$\frac{d^m}{dt^m} t^n = \frac{n!}{(n-m)!} t^{n-m} \tag{12}$$

where $m \leq n$ are positive integers. Note that eq.(12) can be written in terms of the gamma function $\Gamma(n+1) = n!$:

$$\frac{d^m}{dt^m} t^n = \frac{\Gamma(n+1)}{\Gamma(n-m+1)} t^{n-m} \tag{13}$$

Generalizing this by replacing n by k and m by α, where k and α are positive real numbers, we obtain

$$D^{\alpha} t^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha} \tag{14}$$

As an example, we compute the order 1/2 derivative of t:

$$D^{1/2} t = \frac{\Gamma(2)}{\Gamma(3/2)} t^{1/2} = \frac{2}{\sqrt{\pi}} t^{1/2} \tag{15}$$

We note that by the law of exponents of derivatives,

$$D^{1/2} D^{1/2} t = D^{1/2+1/2} t = \frac{d}{dt} t = 1 \tag{16}$$

and we check this by taking the order 1/2 derivative of eq.(15):

$$D^{1/2} \frac{2}{\sqrt{\pi}} t^{1/2} = \frac{2}{\sqrt{\pi}} D^{1/2} t^{1/2} = \frac{2}{\sqrt{\pi}} \frac{\Gamma(3/2)}{\Gamma(1)} t^0 = 1 \tag{17}$$

Now suppose we have a function $x(t)$ which is expandable in a Taylor series about $t = 0$:

$$x(t) = \sum \frac{x^{(k)}(0)}{k!} t^k \tag{18}$$

where $x^{(k)}(0)$ is the k^{th} derivative of x evaluated at $t = 0$. Taking the fractional derivative of both sides,

$$D^{\alpha} x(t) = \sum \frac{x^{(k)}(0)}{k!} D^{\alpha} t^k = \sum \frac{x^{(k)}(0)}{k!} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha} \tag{19}$$

After Ross [20] we note that

$$\int_0^t (t-u)^m u^n du = \frac{m!\,n!}{(m+n+1)!} t^{m+n+1} = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} t^{m+n+1} \tag{20}$$

For example, with $n = 7$ and $m = 3$, macsyma integrates the LHS to be $t^{11}/1320$, and direct evaluation of the RHS coefficient gives $3!\,7!/11! = 1/1320$.
Taking $n = k$ and $m = -1-\alpha$, we get

$$\int_0^t (t-u)^{-1-\alpha} u^k du = \frac{\Gamma(-\alpha)\,\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha} \tag{21}$$

from which we obtain

$$\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha} = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-u)^{-1-\alpha} u^k du \tag{22}$$

Substituting (22) into (19) we obtain

$$D^\alpha x(t) = \sum \frac{x^{(k)}(0)}{k!} \frac{1}{\Gamma(-\alpha)} \int_0^t (t-u)^{-1-\alpha} u^k du \tag{23}$$

Interchanging the processes of summation and integration, we obtain

$$D^\alpha x(t) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-u)^{-1-\alpha} \left\{ \sum \frac{x^{(k)}(0) u^k}{k!} \right\} du \tag{24}$$

which gives, using (18),

$$D^\alpha x(t) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-u)^{-1-\alpha} x(u) du \tag{25}$$

To avoid divergence in eq.(25), we use a trick from Ross [20]. From the law of exponents of derivatives we write

$$D^\alpha x(t) = D^m D^{-p} x(t) \tag{26}$$

where $\alpha = m - p$, where $0 < p < 1$ and where m is the least integer larger than α. Using eq.(25), we obtain

$$D^\alpha x(t) = \frac{d^m}{dt^m} \frac{1}{\Gamma(p)} \int_0^t (t-u)^{p-1} x(u) du \tag{27}$$

In the case that $0 < \alpha < 1$, we have that $m = 1$ and $p = 1-\alpha$, giving

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-u)^{-\alpha} x(u) du \tag{28}$$

For example,

$$D^{1/2} x(t) = \frac{1}{\Gamma(1/2)} \frac{d}{dt} \int_0^t (t-u)^{-1/2} x(u) du \tag{29}$$

As a check we use this formula to compute the order 1/2 derivative of t:

$$D^{1/2} t = \frac{1}{\Gamma(1/2)} \frac{d}{dt} \int_0^t (t-u)^{-1/2} u du = \frac{1}{\Gamma(1/2)} \frac{d}{dt} \left(\frac{4}{3} t^{3/2} \right) = \frac{2}{\sqrt{\pi}} t^{1/2} \tag{30}$$

which agrees with eq.(15). Eq.(28) can be simplified by taking $v = t - u$, giving

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t v^{-\alpha} x(t-v) dv \tag{31}$$

Carrying out the differentiation under the integral sign, we obtain

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t v^{-\alpha} x'(t-v) dv + \frac{x(0)}{t^\alpha} \right) \tag{32}$$

After Ross [20], p.17, we adopt the convention that $x(0) = 0$, giving the final formula:

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t v^{-\alpha} x'(t-v) dv \tag{33}$$

3 MATHIEU'S EQUATION

In this section we present a brief summary of the (non-fractional) Mathieu's equation (10) in order to be able to assess the effects of the addition of a fractional derivative term as in eq.(11). See e.g. Stoker [24]. For given values of the parameters δ and ε, either all solutions of Mathieu's equation are bounded (stable) or an unbounded solution exists (unstable). The $\delta-\varepsilon$ parameter plane is thus divided into stable and unstable regions. Although an infinite number of "resonance tongues" emerge from the δ-axis at $\delta = n^2/4$, where $n = 1, 2, 3, \cdots$, most of these are insignificant for small values of ε, see Fig.1. This is not the case for the tongue emanating from $\delta = 1/4$, which is important in applications and is associated with 2:1 subharmonic resonance. From perturbation theory it is known [19] that this tongue has the following asymptotic expansion:

$$\delta = \frac{1}{4} \pm \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + O(\varepsilon^3) \tag{34}$$

In addition to the aforementioned tongues, there is also a tran-

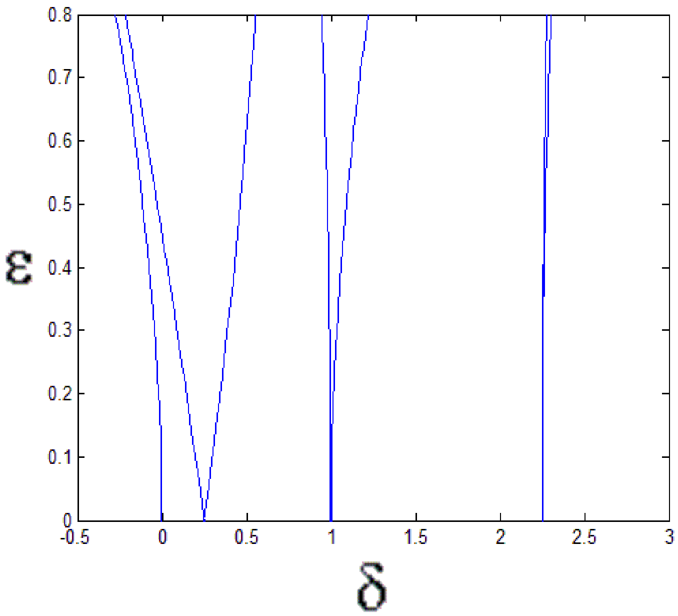

Figure 1. Transition curves in Mathieu's equation (10). Displayed are eqs.(34),(35) as well as other curves whose equations are not listed here. See [19].

sition curve separating stable from unstable regions emanating from the origin in the $\delta - \varepsilon$ plane. It has the following expansion:

$$\delta = -\frac{1}{2}\varepsilon^2 + \frac{7}{32}\varepsilon^4 + O(\varepsilon^6) \tag{35}$$

If a damping term is added, we obtain the damped Mathieu equation:

$$x'' + (\delta + \varepsilon \cos t)x + cx' = 0 \tag{36}$$

The effect of the damping term on the shape of the transition curves is to detach each of the tongues from the δ-axis, thereby requiring a minimum value of ε for instability to occur [19]. In the case of the $n = 1$ tongue, eq.(36) has the following expansion, see Fig.2:

$$\delta = \frac{1}{4} \pm \frac{1}{2}\sqrt{\varepsilon^2 - c^2} + O(\varepsilon^3) \tag{37}$$

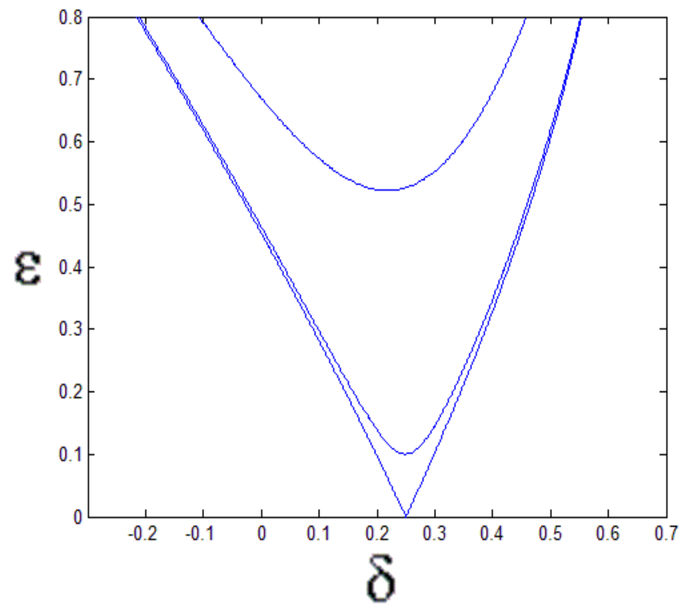

Figure 2. Transition curves (37) in the damped Mathieu equation (36). The upper curve corresponds to $c = 0.5$. The middle curve corresponds to $c = 0.1$. The lower curve corresponds to $c = 0$.

4 FRACTIONAL MATHIEU EQUATION

In this section we use the method of harmonic balance to obtain approximate expressions for the transition curves in the fractional Mathieu equation:

$$x'' + (\delta + \varepsilon \cos t)x + cD^{\alpha}x = 0 \tag{38}$$

From Floquet theory [19] it is known that on the transition curves there exist periodic solutions to (38) with period π or 2π. Thus in order to obtain an approximation for the $n=1$ transition curves, we posit a truncated Fourier series:

$$x = A\cos\frac{t}{2} + B\sin\frac{t}{2} + \cdots \tag{39}$$

In substituting (39) into (38) we need to compute the fractional derivative $D^{\alpha}x$, where $0<\alpha<1$, which we do by using the definition (33):

$$D^{\alpha}x = \frac{1}{\Gamma(1-\alpha)}\int_0^t v^{-\alpha}x'(t-v)dv \tag{40}$$

$$\int_0^t v^{-\alpha}x'(t-v)dv = \frac{1}{2}\int_0^t v^{-\alpha}\left(-A\sin\frac{t-v}{2} + B\cos\frac{t-v}{2}\right)dv \tag{41}$$

Here we use the trig identities:

$$\cos\frac{t-v}{2} = \cos\frac{t}{2}\cos\frac{v}{2} + \sin\frac{t}{2}\sin\frac{v}{2} \tag{42}$$

$$\sin\frac{t-v}{2} = \sin\frac{t}{2}\cos\frac{v}{2} - \cos\frac{t}{2}\sin\frac{v}{2} \tag{43}$$

and eq.(41) becomes

$$\begin{aligned}\int_0^t v^{-\alpha}x'(t-v)dv = {} & \frac{1}{2}\cos\frac{t}{2}\int_0^t v^{-\alpha}\left(B\cos\frac{v}{2} + A\sin\frac{v}{2}\right)dv \\ & + \frac{1}{2}\sin\frac{t}{2}\int_0^t v^{-\alpha}\left(B\sin\frac{v}{2} - A\cos\frac{v}{2}\right)dv \qquad (44) \\ = {} & \frac{1}{2^{\alpha}}\cos\frac{t}{2}\,(BI_c + AI_s) \\ & + \frac{1}{2^{\alpha}}\sin\frac{t}{2}\,(-AI_c + BI_s) \qquad (45)\end{aligned}$$

where

$$I_c = \int_0^{t/2} w^{-\alpha}\cos w\,dw \quad \text{and} \tag{46}$$

$$I_s = \int_0^{t/2} w^{-\alpha}\sin w\,dw \tag{47}$$

Although these integrals cannot be evaluated in closed form for general values of t, maxima is able to evaluate them in the limit as $t\to\infty$:

$$\int_0^{\infty} w^{-\alpha}\cos w\,dw = \Gamma(1-\alpha)\sin\frac{\alpha\pi}{2} \quad \text{and} \tag{48}$$

$$\int_0^{\infty} w^{-\alpha}\sin w\,dw = \Gamma(1-\alpha)\cos\frac{\alpha\pi}{2} \tag{49}$$

After [28], [27] we approximate I_c and I_s in eqs.(45),(46),(47), by their values in eqs.(48),(49) in what follows, thereby restricting attention to the large t limit. Thus we find from eq.(40) the following expression for the fractional derivative:

$$\begin{aligned}D^{\alpha}x = {} & \frac{1}{2^{\alpha}}\cos\frac{t}{2}\,(B\sin\frac{\alpha\pi}{2} + A\cos\frac{\alpha\pi}{2}) + \\ & \frac{1}{2^{\alpha}}\sin\frac{t}{2}\,(-A\sin\frac{\alpha\pi}{2} + B\cos\frac{\alpha\pi}{2})\end{aligned} \tag{50}$$

Next we substitute (39) and (50) into (38) and collect terms, equating to zero coefficients of $\sin\frac{t}{2}$ and $\cos\frac{t}{2}$. Eliminating A and B from the resulting two equations gives the following approximate expression for the $n=1$ transition curves:

$$\delta = \frac{1}{4} - \frac{c}{2^{\alpha}}\cos\frac{\alpha\pi}{2} \pm \frac{\sqrt{2^{2\alpha}\varepsilon^2 - 4c^2\sin^2\frac{\alpha\pi}{2}}}{2^{\alpha+1}} \tag{51}$$

As a check, we substitute $\alpha=1$ in eq.(51), in which case the fractional derivative in eq.(38) becomes an ordinary first derivative and we obtain eq.(37) corresponding to the damped Mathieu equation (36). Fig.3 displays the transition curves (51) for various values of α.

We may obtain a higher order approximation by replacing the original ansatz (39) by the following:

$$x = A\cos\frac{t}{2} + B\sin\frac{t}{2} + G\cos\frac{3t}{2} + H\sin\frac{3t}{2} + \cdots \tag{52}$$

Proceeding as before we obtain an algebraic equation relating δ, ε and α which is too complicated to list here. See Fig.4 where it is displayed along with eq.(51) for $\alpha = 1/2$.

In a similar fashion we may obtain approximations for the other transition curves. For example, in order to obtain an expression for the $n=0$ transition curve which passes through the origin in the $\delta-\varepsilon$ plane, we start with the ansatz:

$$x = A\cos t + B\sin t + G + \cdots \tag{53}$$

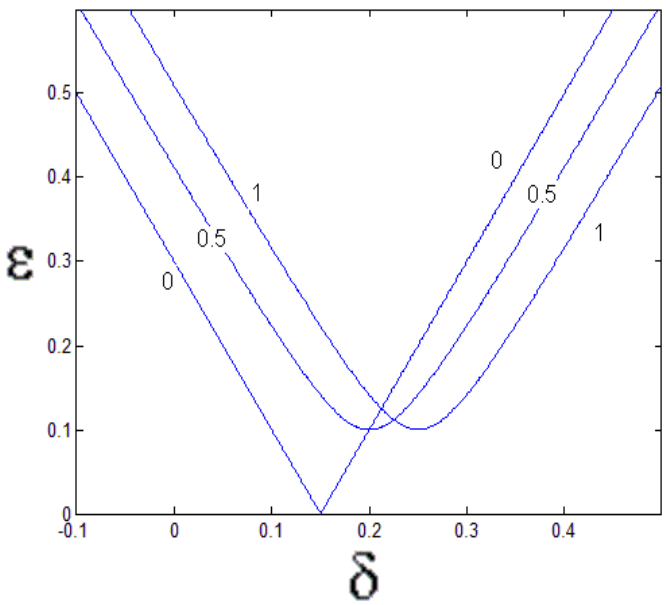

Figure 3. $n = 1$ transition curve, eq.(51), in the fractional Mathieu equation (38) for $c = 0.1$ and $\alpha = 0, 0.5, 1$.

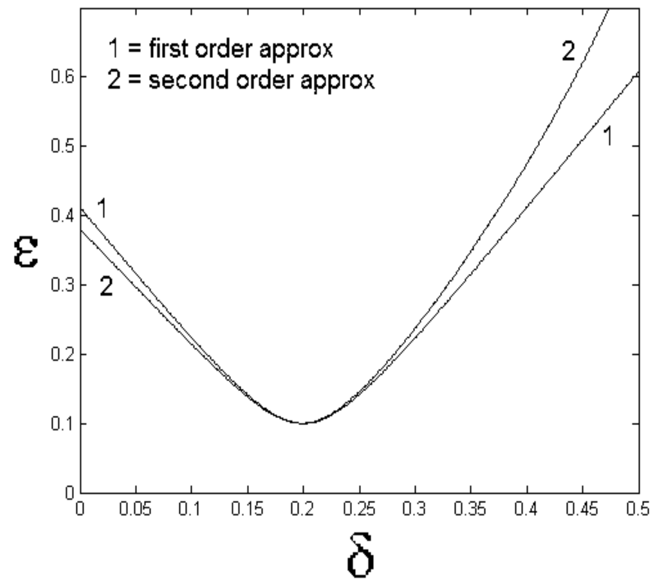

Figure 4. Transition curves in the fractional Mathieu equation (38) for $\alpha = 0.5$ and $c = 0.1$. First and second order approximations, as obtained by the method of harmonic balance. The first order approximation is given by eq.(51). The second order approximation has 51 terms and is too long to list here.

Proceeding as before we obtain the following expression for the $n = 0$ transition curve:

$$(Kc-1)\varepsilon^2 + (\varepsilon^2 - 2c^2 + 4Kc - 2)\delta + 4(1-Kc)\delta^2 - 2\delta^3 = 0 \tag{54}$$

where $K = \cos\frac{\alpha\pi}{2}$. Fig.5 shows the $n = 0$ transition curve for various values of α. We note that the shape of the transition curve does not change very much for α in the range [0,1].

5 DISCUSSION

In the case of the $n = 1$ transition curves (see eq.(51) and Fig.3), we see that a change in the order α of the fractional derivative affects the shape and location of the transition curves. This effect can be characterized by the location of the lowest point on the transition curve, which represents the minimum quantity of forcing amplitude ε necessary to produce instability. Let us refer to this minimum value of ε, for a given value of α, as ε_{min}. See Fig.6, where eq.(51) is displayed as a surface in $\delta-\varepsilon-\alpha$ space.

In order to obtain an expression for ε_{min}, we may differentiate eq.(51) with respect to ε, giving the slope of the transition curve, and require this slope to be infinite. We find:

$$\varepsilon_{min} = 2c\left(\frac{\sin\frac{\alpha\pi}{2}}{2^\alpha}\right) \tag{55}$$

See Fig.7, where ε_{min} is plotted as a function of α. The greatest effect is observed where this curve achieves its maximum, shown by a dot in Fig.7. Let us refer to the corresponding value of α as α^*. Then we may obtain an expression for α^* by differentiating eq.(55) with respect to α and setting $d\varepsilon_{min}/d\alpha$ equal to zero. We find:

$$\alpha^* = \frac{2}{\pi}\arctan\frac{\pi}{2\log 2} \approx 0.735 \tag{56}$$

Let us refer to the corresponding value of ε_{min} as ε^*_{min}. We find:

$$\varepsilon^*_{min} \approx 1.099c \tag{57}$$

This effect is reminiscent of the (non-fractional) damped Mathieu eq.(36), cf.Fig.2, which corresponds here to $\alpha = 1$. Note from Fig.7, that when α lies in the range (0.5,1), the values for ε_{min} are all greater than ε_{min} for eq.(36). Thus we may say that the damping effect of the fractional derivative term in eq.(38), for $0.5 < \alpha < 1$, is greater than that of the (non-fractional) damped Mathieu eq.(36).

Note also that in contrast to non-fractional damping, fractional damping also moves this lowest point on the transition curve in a horizontal direction, see Fig.3, thereby effectively changing the resonant value of δ.

On the other hand, in the case of the $n = 0$ transition curves (see eq.(54) and Fig.5), we see that there is very little change as α is varied.

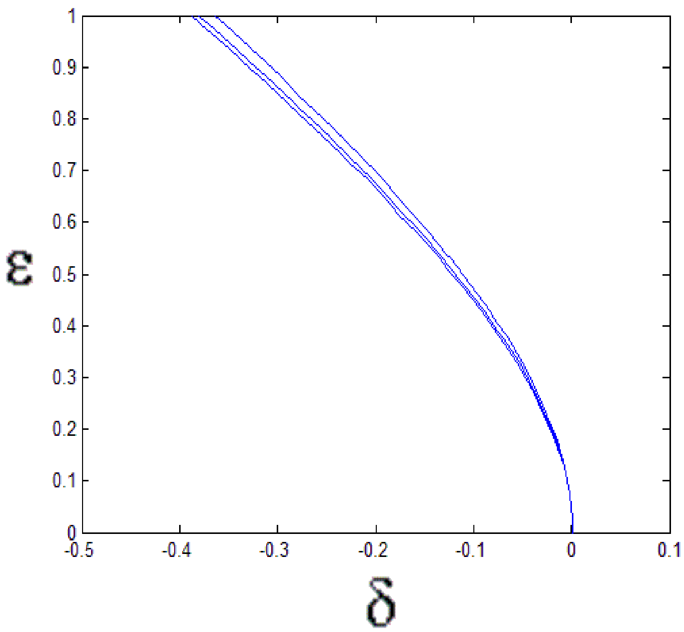

Figure 5. $n = 0$ transition curve, eq.(54), in the fractional Mathieu equation (38) for $c = 0.1$ and $\alpha = 0, 0.5, 1$. The leftmost curve corresponds to $\alpha = 0$. The middle curve corresponds to $\alpha = 0.5$. The rightmost curve corresponds to $\alpha = 1$.

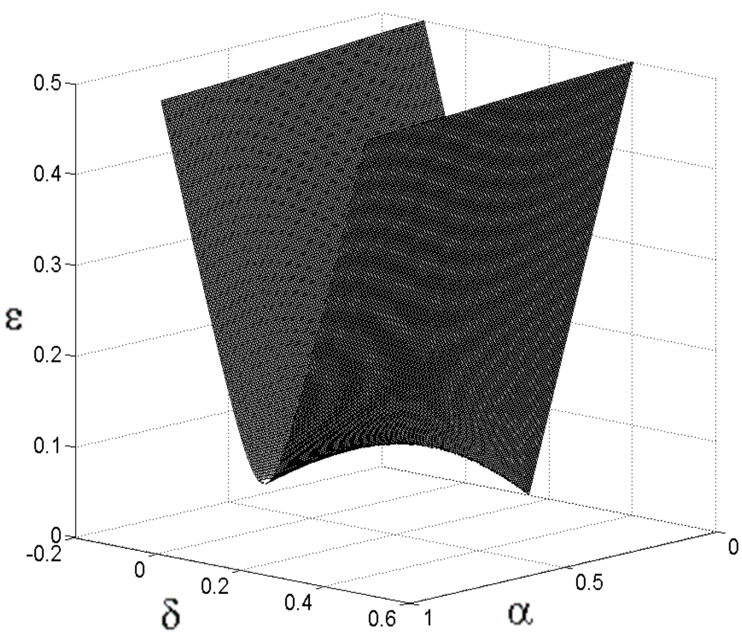

Figure 6. Eq.(51) displayed in $\delta - \varepsilon - \alpha$ space for $c = 0.1$

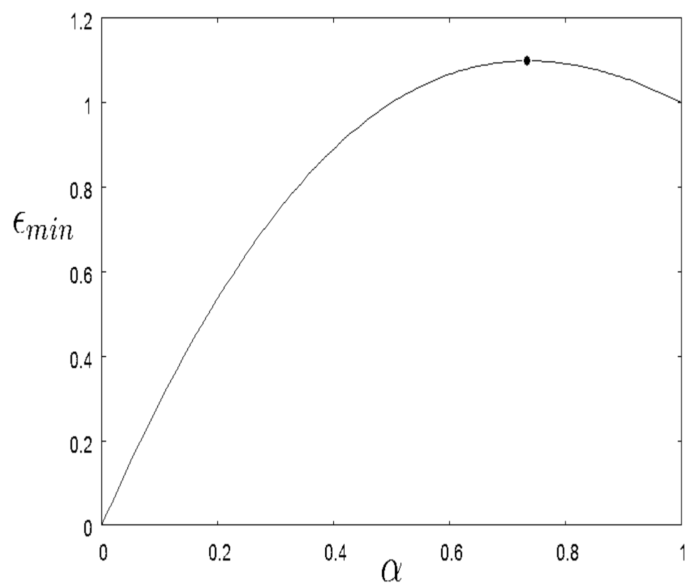

Figure 7. Plot of ε_{min}, the minimum quantity of forcing amplitude ε necessary to produce instability, as a function of fractional derivative order α, eq.(55). The greatest effect is observed where this curve achieves its maximum, shown as a dot here, and referred to as α^* in the text.

6 CONCLUSION

In this paper we have used the method of harmonic balance to obtain explicit approximate expressions for the $n = 1$ and $n = 0$ transition curves separating regions of stability from regions of instability in the fractional Mathieu equation (38). We showed that by changing the value of the order of the fractional derivative, α, the shape and location of the $n = 1$ transition curve can be changed. In particular we showed that the minimum quantity of forcing amplitude ε necessary to produce instability was greatest for $\alpha^* \approx 0.735$.

This work represents a first step in developing a complete theory of fractional parametric excitation. Related work that lies ahead could include the effects of phenomena that have been applied to non-fractional Mathieu equations, such as nonlinearity [19], quasiperiodic forcing [30], delay [13], partial differential equations [18] and slow passage through resonance [3].

7 ACKNOWLEDGEMENTS

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