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DISAPPEARANCE OF RESONANCE TONGUES

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ABSTRACT

We investigate a phenomenon observed in systems of the form

$$dx/dt = a_1(t)x + a_2(t)y$$
$$dy/dt = a_3(t)x + a_4(t)y$$

where

$$a_i(t) = P_i + \varepsilon Q_i \cos 2t$$

where P_i, Q_i and ε are given constants, and where it is assumed that when ε=0 this system exhibits a pair of linearly independent solutions of period 2π. Since the driver $\cos 2t$ has period π, we have the ingredients for a 2:1 subharmonic resonance which typically results in a tongue of instability involving unbounded solutions when $\varepsilon > 0$. We present conditions on the coefficients P_i, Q_i such that the expected instability does not occur, i.e., the tongue of instability has disappeared.

INTRODUCTION

This paper concerns *parametric resonance,* which may be described as a 2:1 subharmonic resonance commonly occurring in systems of O.D.E.'s which involve periodic coefficients. The paradigm example is given by Mathieu's equation,

$$\frac{d^2x}{dt^2} + (\delta + \varepsilon \cos 2t)x = 0 \tag{1}$$

When $\delta = 1$ and $\varepsilon = 0$ eq.(1) exhibits a periodic solution of period 2π. When δ is close to 1 and $\varepsilon > 0$, eq.(1) exhibits a tongue of instability in the δ-ε parameter plane, see Fig.1. A perturbation analysis valid for $\varepsilon << 1$ gives the equation of the tongue in the form [1], [2], [3]:

$$\delta = 1 \pm \frac{\varepsilon}{2} + O(\varepsilon^2) \tag{2}$$

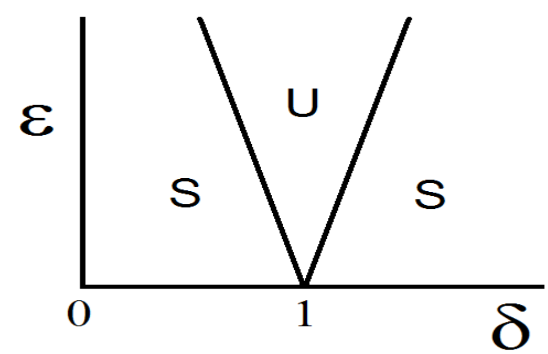

FIGURE 1. 2:1 subharmonic resonance tongue for eq.(1). S=stable (bounded in time). U=unstable (unbounded in time).

Parameters inside the tongue correspond to equations which include solutions which are unbounded in time, while points outside the tongue correspond to equations for which solutions are bounded and quasiperiodic. Equations with solutions that include periodic motions of period 2π (i.e. with period twice the period of the forcing function $\cos 2t$, which has period π) correspond to points on the tongue boundaries.

If eq.(1) were to be generalized to include a third parameter μ, then the shape of the tongue would in general be expected to depend on μ. For example, writing eq.(1) as a system of two first order O.D.E.'s,

$$\frac{dx}{dt} = y \tag{3}$$
$$\frac{dy}{dt} = -(\delta + \varepsilon \cos 2t)x \tag{4}$$

we could include an additional forcing term as follows:

$$\frac{dx}{dt} = y + \varepsilon \mu x \cos 2t \tag{5}$$
$$\frac{dy}{dt} = -(\delta + \varepsilon \cos 2t)x \tag{6}$$

Now when μ =0, eqs.(5),(6) reduce to eqs.(3),(4) and the tongue is as in Fig.1. However for nonzero μ, a perturbation analysis valid for $\varepsilon <<1$ gives the equation of the tongue in the form:

$$\delta = 1 \pm \frac{\varepsilon}{2}\sqrt{1+\mu^2} + O(\varepsilon^2) \tag{7}$$

Note that for this example, although the shape of the tongue depends on μ, there is no (real) value of μ for which the tongue closes up and disappears to O(ε^2).

In this paper we investigate the phenomenon in which a resonant tongue closes up and disappears as a parameter is varied. Our motivation comes from an example, presented in the next section, which occurred incidentally in an application [4]. Previous work in this area includes treatment of problems in which an expected resonant tongue is absent. An example system exhibiting such behavior is Ince's equation:

$$(1 + a\cos 2t)\frac{d^2x}{dt^2} + b\sin 2t\frac{dx}{dt} + (c + d\cos 2t)x = 0 \tag{8}$$

This equation was treated by Magnus and Winkler [5] using a method which is unrelated to that used in the present paper. Their work was extended by Recktenwald and Rand [6], [1].

EXAMPLE

Now let us consider the following example, which actually arose in the study of an evolutionary dynamics problem [4]:

$$\frac{dx}{dt} = \frac{(-1+2(\mu-1)\varepsilon\cos 2t)x + (-2+(2+\mu)\varepsilon\cos 2t)y}{\sqrt{3}\,\delta} \tag{9}$$
$$\frac{dy}{dt} = \frac{(2+(1-4\mu)\varepsilon\cos 2t)x + (1-(1+2\mu)\varepsilon\cos 2t)y}{\sqrt{3}\,\delta} \tag{10}$$

For δ =1 and ε=0, eqs.(9) and (10) take the form

$$\sqrt{3}\,\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{11}$$

Eqs.(11) have the general solution:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} \sqrt{3}\cos t - \sin t \\ 2\sin t \end{pmatrix} + B\begin{pmatrix} -2\sin t \\ \sqrt{3}\cos t + \sin t \end{pmatrix} \tag{12}$$

where A and B are arbitrary constants. Since the solution (12) has period 2π and the forcing function in eqs.(11) has period π, we again have an instance of 2:1 subharmonic forcing in a parametrically driven system, and we are not surprised to find a resonance tongue in the δ-ε plane emanating from δ=1, ε=0. As shown in [4], perturbation analysis valid for $\varepsilon <<1$ gives the equation of the tongue in the form:

$$\delta = 1 \pm \frac{1}{2}(\mu - 1)\varepsilon + O(\varepsilon^2) \tag{13}$$

In particular we see that when μ=1, the tongue closes up to O(ε^2). The question is, is this an approximate result, valid only to some order of ε, or is it exact, valid to all orders of ε?

A natural way to check this result is to use numerical integration together with Floquet theory [1] to generate the tongue boundaries for various values of μ. This is done in Fig.2, where we see that the tongue region does indeed appear to be getting smaller as μ approaches unity. Nevertheless, numerical integration is an approximate process, and this result still leaves doubt as to whether the tongue has truly closed up. In the next section we address this question by examining the behavior of solutions on the line δ=1 for μ=1. If the tongue really did close up, then this line would represent the two tongue boundaries as they press together and coalesce. If all solutions are bounded on this line, then we could conclude that no instability is present, and that the tongue has vanished.

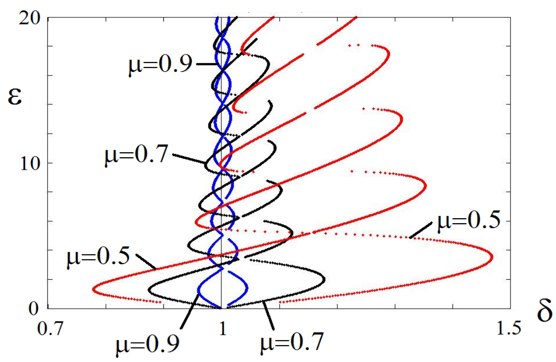

FIGURE 2. 2:1 subharmonic resonance tongues for eqs.(9) and (10) for μ=0.5,0.7,0.9.

THEOREM

In this section we generalize the preceding example by considering a system in the following form:

$$\frac{dx}{dt} = a_1(t)\,x + a_2(t)\,y \tag{14}$$
$$\frac{dy}{dt} = a_3(t)\,x + a_4(t)\,y \tag{15}$$

where

$$a_i(t) = P_i + \varepsilon\, Q_i \cos 2t \tag{16}$$

We assume that when ε=0, the system (14),(15) has a pair of linearly independent solutions of period 2π (cf. eqs.(12)), which we write in the form:

$$x = A\, F_1(t) + B\, F_2(t) \tag{17}$$
$$y = A\, G_1(t) + B\, G_2(t) \tag{18}$$

where A and B are arbitrary constants.

The goal is then to find conditions on the coefficients P_i and Q_i such that eqs.(14),(15) have two linearly independent solutions of period 2π *for all* $\varepsilon > 0$. The coexistence of these two solutions for all $\varepsilon > 0$ means that the associated 2:1 resonance tongue has closed up, both boundaries being coincident. (In general each boundary of the tongue possesses a bounded period 2π solution. If the system (14),(15) possesses two linearly independent period 2π solutions, then these solutions are a basis for the solution space and all solutions are bounded: both tongue boundaries are then coincident, and the tongue has closed up.)

When ε=0, eqs.(14),(15) become

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{19}$$

For eqs.(19) to exhibit two linearly independent solutions of period 2π, its eigenvalues must be $\pm i$. This requires that the trace of the matrix in (19) be zero, and its determinant be unity, giving:

$$P_4 = -P_1 \tag{20}$$
$$P_3 = \frac{-1 - P_1^2}{P_2} \tag{21}$$

Then without loss of generality we may take the two linearly independent period 2π solutions in (17),(18) to be:

$$F_1(t) = \sin t, \qquad G_1(t) = \nu_1 \sin t + \nu_2 \cos t \tag{22}$$

and

$$G_2(t) = \sin t, \qquad F_2(t) = \nu_3 \sin t + \nu_4 \cos t \tag{23}$$

where the ν_i coefficients may be found by substituting (22),(23) into the ε=0 equations (19), giving:

$$\nu_1 = -\frac{P_1}{P_2} \tag{24}$$
$$\nu_2 = \frac{1}{P_2} \tag{25}$$
$$\nu_3 = -\frac{P_1 P_2}{1 + P_1^2} \tag{26}$$
$$\nu_4 = -\frac{P_2}{1 + P_1^2} \tag{27}$$

Having characterized the solution of eqs.(14),(15) for ε=0, we now go after the solution for $\varepsilon > 0$.

We posit a solution for $\varepsilon > 0$ using variation of parameters (cf. eqs.(17),(18)):

$$x = u(t)\, F_1(t) + v(t)\, F_2(t) \tag{28}$$
$$y = u(t)\, G_1(t) + v(t)\, G_2(t) \tag{29}$$

where $u(t)$ and $v(t)$ are unknown functions to be found. Substituting (28),(29) into (14),(15), and solving for du/dt and dv/dt, we get

$$\frac{du}{dt} = \varepsilon \cos 2t \left[H_1 \sin^2 t + H_2 \sin t \cos t + H_3\right] \tag{30}$$

$$\frac{dv}{dt} = \varepsilon \cos 2t \left[H_4 \sin^2 t + H_5 \sin t \cos t + H_6\right] \tag{31}$$

where $H_i = H_i(P_1, P_2, Q_i, u, v)$ are known functions, too long to list here.

Motivated by a desire to find conditions which guarantee a pair of linearly independent solutions to eqs.(14),(15), we set $H_1 = H_2 = H_4 = H_5 = 0$, which requires

$$Q_2 = -\frac{P_2}{2P_1}(Q_4 - Q_1) \tag{32}$$

$$Q_3 = \frac{(1+P_1^2)}{2P_1P_2}(Q_4 - Q_1) \tag{33}$$

Assuming that conditions (32),(33) are fulfilled, eqs.(30),(31) become:

$$\frac{du}{dt} = \varepsilon \cos 2t \left[Q_4\, u - \frac{P_2}{2P_1}(Q_4 - Q_1)\, v\right] \tag{34}$$

$$\frac{dv}{dt} = \varepsilon \cos 2t \left[\frac{1+P_1^2}{2P_1P_2}(Q_4 - Q_1)\, u + Q_1\, v\right] \tag{35}$$

Next we reparameterize t by using a new time scale τ defined as:

$$\tau = \frac{1}{2}\sin 2t \tag{36}$$

which transforms (34),(35) into a system with constant coefficients:

$$\frac{du}{d\tau} = \varepsilon \left[Q_4\, u - \frac{P_2}{2P_1}(Q_4 - Q_1)\, v\right] \tag{37}$$

$$\frac{dv}{d\tau} = \varepsilon \left[\frac{1+P_1^2}{2P_1P_2}(Q_4 - Q_1)\, u + Q_1\, v\right] \tag{38}$$

Although the solution of eqs.(37),(38) depends on the numerical values of the coefficients P_1, P_2, Q_1, Q_4, it will in general consist of sinusoidal and or exponential functions of τ, and since τ is π-periodic in t by eq.(36), $u(t)$ and $v(t)$, the solution functions of (37),(38), will be π-periodic in t. Furthermore, we note by eqs.(28),(29), that $x(t)$ and $y(t)$, the solution functions of eqs.(14),(15), are composed of terms which are the product of π-periodic functions (i.e. $u(t)$ and $v(t)$) and 2π-periodic functions (i.e. $F_i(t)$ and $G_i(t)$). Now since the product of a π-periodic function and a 2π-periodic function has period 2π, we see that all solutions of eqs.(14),(15) are 2π periodic.

We have therefore proved the following

THEOREM: All nontrivial solutions of eqs.(14),(15) will have period 2π if eqs.(20),(21),(32),(33) are satisfied.

APPLICATION

In this section we apply the theorem of the previous section to the example presented earlier in this paper. Eqs.(9),(10) with δ=1, when expressed in the form of eqs.(14),(15), yield the following values for the coefficients P_i and Q_i:

$$P_1 = -\frac{1}{\sqrt{3}} \tag{39}$$

$$P_2 = -\frac{2}{\sqrt{3}} \tag{40}$$

$$P_3 = \frac{2}{\sqrt{3}} \tag{41}$$

$$P_4 = \frac{1}{\sqrt{3}} \tag{42}$$

$$Q_1 = \frac{-2+2\mu}{\sqrt{3}} \tag{43}$$

$$Q_2 = \frac{2+\mu}{\sqrt{3}} \tag{44}$$

$$Q_3 = \frac{1-4\mu}{\sqrt{3}} \tag{45}$$

$$Q_4 = -\frac{1+2\mu}{\sqrt{3}} \tag{46}$$

Inspection shows that these values for P_i and Q_i satisfy eqs.(20),(21),(32),(33) in the case that μ=1. Therefore we may conclude by the foregoing theorem that the closing of the tongue in eqs.(9),(10) at μ=1 is an exact result and is valid to all orders of ε.

ANOTHER EXAMPLE

As an additional check on the theorem, in this section we try to invent a simple example which satisfies the requirements of the theorem, namely eqs.(20),(21),(32),(33), and then use numerical integration to confirm the disappearance of the associated tongue. The equations which must be satisfied are:

$$P_4 = -P_1 \tag{47}$$

$$P_3 = \frac{-1-P_1^2}{P_2} \tag{48}$$

$$Q_2 = -\frac{P_2}{2P_1}(Q_4 - Q_1) \tag{49}$$

$$Q_3 = \frac{(1+P_1^2)}{2P_1P_2}(Q_4 - Q_1) \tag{50}$$

Since we are seeking a simple example, we choose to satisfy eqs.(49),(50) by taking Q_4=Q_1=1, which gives Q_2=Q_3=0. Then to satisfy eq.(47) simply, we choose P_1=1, which gives P_4=−1. Then eq.(48) becomes $P_3=-2/P_2$, which prompts us to choose P_2=1 and P_3=−2. The resulting example takes the form (cf. eqs.(14)-(16)):

$$\frac{dx}{dt} = (1+\varepsilon\cos 2t)x + y \tag{51}$$

$$\frac{dy}{dt} = -2x + (-1+\varepsilon\cos 2t)y \tag{52}$$

So according to the theorem presented above, all solutions to eqs.(51) and (52) are bounded periodic functions with period 2π. Now we need a parameter δ so that we can detune from the 2:1 resonance and get a tongue. We choose to replace the two unit coefficients in eqs.(51) and (52) by $1+\delta$, giving:

$$\frac{dx}{dt} = (1+\delta+\varepsilon\cos 2t)x + y \tag{53}$$

$$\frac{dy}{dt} = -2x + (-(1+\delta)+\varepsilon\cos 2t)y \tag{54}$$

Thus we may expect eqs.(53) and (54) to exhibit a degenerate tongue (one that is closed up) in the δ-ε plane along the line δ=0. Finally we need another parameter, μ, which allows the tongue to open up, as in our original system eqs.(9) and (10). We choose to replace the ε coefficient in eq.(53) by $\varepsilon\mu$, giving the final form of our synthesized example:

$$\frac{dx}{dt} = (1+\delta+\varepsilon\mu\cos 2t)x + y \tag{55}$$

$$\frac{dy}{dt} = -2x + (-(1+\delta)+\varepsilon\cos 2t)y \tag{56}$$

Thus this system should have a resonance tongue in the δ-ε plane which emanates from the point $\delta=0$, ε=0, and which closes up and disappears as μ approaches unity.

To check this result, we again use numerical integration together with Floquet theory [1] to generate the tongue boundaries for various values of μ. This is done in Figs.3 and 4, where we see that the tongue region does indeed appear to be closing up as μ approaches unity.

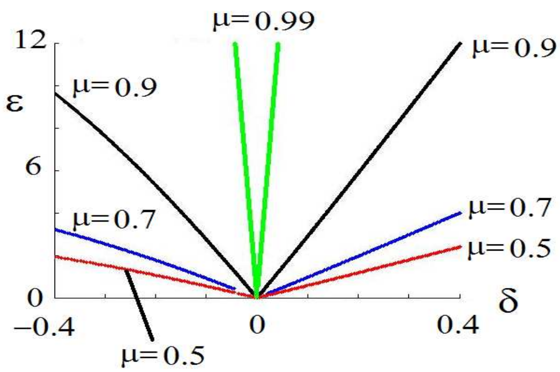

FIGURE 3. 2:1 subharmonic resonance tongues for eqs.(55) and (56) for μ=0.5,0.7,0.9,0.99.

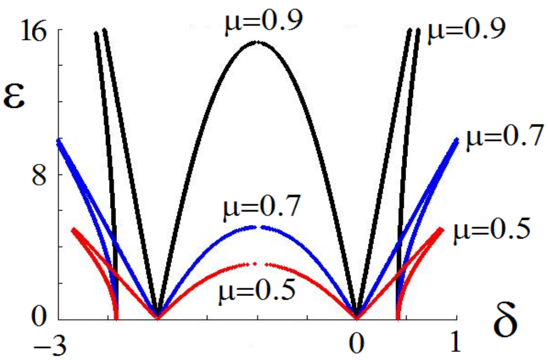

FIGURE 4. 2:1 subharmonic resonance tongues for eqs.(55) and (56) for μ=0.5,0.7,0.9. Figure 3 is an expanded view of the region around $\delta = 0$.

CONCLUSION

We have investigated the dynamics of a class of linear parametrically excited systems of the form

$$dx/dt = a_1(t)x + a_2(t)y$$
$$dy/dt = a_3(t)x + a_4(t)y$$

where

$$a_i(t) = P_i + \varepsilon Q_i \cos 2t$$

We presented a theorem which guarantees that all solutions of such a system are bounded 2π periodic functions of time if certain conditions on the parameters are satisfied. In particular we require that the ε=0 system possesses two linearly independent period 2π solutions, which requires that the P_i coefficients satisfy the conditions:

$$P_4 = -P_1$$

$$P_3 = (-1-P_1^2)/P_2$$

For all solutions of the ε non-zero system to be 2π periodic, we additionally require that

$$Q_2 = -P_2(Q_4 - Q_1)/(2P_1)$$
$$Q_3 = (1+P_1^2)(Q_4 - Q_1)/(2P_1P_2)$$

Note that this theorem applies for any value of ε, i.e., it does not require that ε be small as in the case of a perturbation method.

The theorem was applied to systems in which a resonant tongue of instability disappears as a parameter is varied. Although perturbation methods and numerical integration might suggest that such a disappearance has occurred, it requires a rigorous proof to be certain. The theorem presented in this paper provides this proof by showing that the instability associated with the resonant tongue is absent, replaced by bounded periodic motion.

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