Proceedings of the ASME 2013 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference
IDETC/CIE 2013
August 4-7, 2013, Portland, Oregon, USA

DETC2013-12046

ALTERNATE MODELS OF REPLICATOR DYNAMICS

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ABSTRACT

Models of evolutionary dynamics are often approached via the replicator equation, which in its standard form is given by

$$\dot{x}_i = x_i\left(f_i(x) - \phi\right),$$

$i = 1,\ldots,n$, where x_i is the frequency, or relative abundance, of strategy i, f_i is its fitness, and $\phi = \sum_{i=1}^{n} x_i f_i$ is the average fitness. A game-theoretic aspect is introduced to the model via the payoff matrix A, where $A_{i,j}$ is the expected payoff of i vs. j, by taking $f_i(x) = (A\cdot x)_i$. This model is based on the exponential model of population growth, $\dot{x}_i = x_i f_i$, with ϕ introduced in order both to hold the total population constant and to model competition between strategies. We analyze the dynamics of analogous models for the replicator equation of the form

$$\dot{x}_i = g(x_i)\left(f_i - \phi\right),$$

for selected growth functions g.

INTRODUCTION

The field of evolutionary dynamics combines game theory with ordinary differential equations to model Darwinian evolution via competition between adaptive strategies. A common approach [1] uses the replicator equation, which modifies the exponential model of population growth, $\dot{x}_i = x_i f_i$, where f_i is the fitness of strategy i, by introducing the average fitness over all strategies, ϕ. The change in the relative abundance, x_i, is then

$$\dot{x}_i = x_i(f_i - \phi), \tag{1}$$

where ϕ is chosen so that $\{x \in R^n : \sum x_i = 1, 0 \le x_i \le 1\}$ is an invariant manifold. This means that $\sum \dot{x}_i = 0$, so

$$\phi = \frac{\sum x_i f_i}{\sum x_i} = \sum x_i f_i. \tag{2}$$

In essence, ϕ acts as a coupling term that introduces dependence on the abundance and fitness of other strategies.

In this work, we generalize the replicator model by replacing the base model $\dot{x}_i = x_i f_i$ by $\dot{x}_i = g(x_i) f_i$, where g is a natural growth function. The replicator equation for each strategy becomes

$$\dot{x}_i = g(x_i)(f_i - \phi), \tag{3}$$

where ϕ is now a modified average fitness, again chosen so that $\sum x_i = 1$.

The game-theoretic component of this model lies in the choice of fitness functions. Take the payoff matrix A, whose (i,j)-th entry is the expected reward for strategy i when it competes with strategy j. The fitness f_i of strategy i is then $(A\cdot x)_i$, where $x \in R^n$ is the vector of frequencies x_i. In this work, we use a payoff matrix representing a game analogous to rock-paper-scissors (RPS): there are three strategies, each of which has an advantage versus one other and a disadvantage versus the third. Each strategy is neutral versus itself.

Analysis of the resulting dynamical system is presented. We find that for the logistic model

$$g(x) = x - ax^2, \tag{4}$$

with appropriate choices of the parameter a, there are multiple fixed points of the system that do not exist in the usual model $g(x) = x$. We will show that when A is chosen so that the RPS game is zero-sum, there are 13 equilibria: one neutrally stable equilibrium with all three strategies surviving; three saddle points with all three strategies surviving; three saddles with only one surviving strategy; and three attracting and three repelling fixed points where two strategies survive. The system exhibits both periodic motion and convergence to attractors. We analyze the symmetries of this system, and its bifurcations as the entries of A vary.

This alternate formulation may be useful in modeling natural or social systems that are not adequately described by the usual replicator dynamics.

DERIVATION

Let us review the usual replicator dynamics. We have $f_i = f_i(x)$, where $f_i(x) = (A \cdot x)_i$, where A is the payoff matrix. The average payoff is thus $\phi = \sum_i x_i f_i$, and the change in frequency of strategy i is given by the product of the frequency x_i and its payoff relative to the average. In this model, all population-dependence of the effectiveness (hence growth rate) of strategy i is accounted for by f_i. However, we wish our fitness functions f_i to represent the game-theoretic payoff of individual-level competition. We therefore include some of the population dependence in a growth function $g(x_i)$; this represents the growth rate of the raw population using strategy i, in the absence of competition. Thus the expected population-level payoff of strategy i is $g(x_i) f_i$, and the average population-level payoff is

$$\phi = \frac{\sum_i g(x_i) f_i}{\sum_i g(x_i)}. \tag{5}$$

We require that in this model, ϕ (and hence $\dot{x}$) is only defined for growth functions g such that the denominator does not vanish for any x in the region of interest. With that caveat, using this definition of ϕ, the replicator equation becomes

$$\dot{x}_i = g(x_i)(f_i - \phi), \; i = 1, \ldots, n. \tag{6}$$

We can verify that

$$\begin{aligned}
\sum_i \dot{x}_i &= \sum_i g(x_i)(f_i - \phi) \\
&= \sum_i g(x_i) f_i - \sum_i g(x_i) \frac{\sum_i g(x_i) f_i}{\sum_i g(x_i)} \\
&= 0
\end{aligned} \tag{7}$$

so the total population over all strategies is constant, and it is valid to say that each x_i represents the *frequency* of strategy i. We will use the term *relative abundance* for x_i whenever there is ambiguity between x_i and the time-frequency of any periodic motion in the dynamics.

ROCK-PAPER-SCISSORS

We consider the game-theoretic case in which $n = 3$ and f_i is given by $f_i(x) = (A \cdot x)_i$, where A is the payoff matrix

$$A = \begin{pmatrix} 0 & -1 & +1 \\ +1 & 0 & -1 \\ -1 & +1 & 0 \end{pmatrix}, \tag{8}$$

representing a zero-sum rock-paper-scissors game. That is, writing (x_1, x_2, x_3) as (x, y, z),

$$f_1 = z - y, \quad f_2 = x - z, \quad f_3 = y - x. \tag{9}$$

We note that this model has been shown to be relevant to biological applications [2], [3], and to social interactions [4]. Note that the dynamics of the 3-strategy game takes place on the triangle in R^3 (in fact, the three-dimensional simplex)

$$\Sigma = \left\{ (x,y,z) \in R^3 : x+y+z = 1 \text{ and } x,y,z \geq 0 \right\}. \tag{10}$$

Therefore we can eliminate z using $z = 1 - x - y$. This reduces the problem to two dimensions, so that (6) becomes

$$\dot{x} = \frac{g(x)\left((1-3y)\, g(1-x-y) + (2-3x-3y)\, g(y)\right)}{g(x) + g(y) + g(1-x-y)} \tag{11}$$

$$\dot{y} = -\frac{g(y)\left((1-3x)\, g(x) + (2-3x-3y)\, g(1-x-y)\right)}{g(x) + g(y) + g(1-x-y)} \tag{12}$$

where we have used ϕ as defined in Eqn (5). This vector field is defined on the projection of Σ onto the $x-y$ plane. We will refer to this region as

$$T = \{(x,y) : (x, y, 1-x-y) \in \Sigma\}. \tag{13}$$

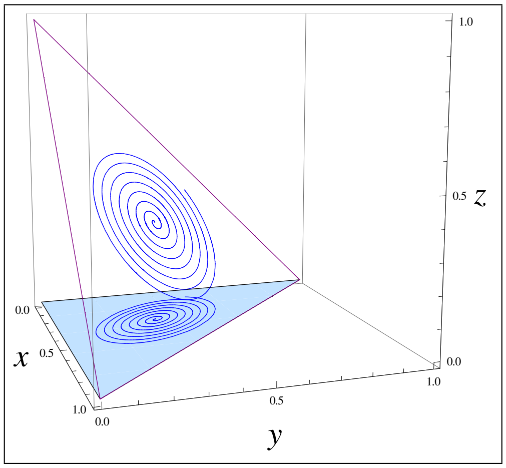

FIGURE 1. A curve in Σ and its projection in T.

Note that since Σ, the region of interest for the three-dimensional flow, is confined to a plane in R^3, the projection down to T loses no information. See Fig. 1.

CHOICES OF GROWTH FUNCTION

Taking $g(x_i) = x_i/(1+ax_i)$

First, consider the case where the growth function is given by $g(x_i) = \frac{x_i}{1+ax_i}$. This growth function increases monotonically in x_i, leading to dynamics that are qualitatively similar to the standard $g(x_i) = x_i$ case. We find that Eqns. (11) and (12) become

$$\dot{x} = \frac{-x(-1+x+2y)(-1+3ay(-1+x+y)}{-1+3a^2xy(-1+x+y)+2a(x^2+x(-1+y)+(-1+y)y))} \tag{14}$$

$$\dot{y} = \frac{y(-1+2x+y)(-1+3ax(-1+x+y))}{-1+3a^2xy(-1+x+y)+2a(x^2+x(-1+y)+(-1+y)y)} \tag{15}$$

Solving $\dot{x} = \dot{y} = 0$, we find that the equilibria are located at the corners of T,

$$(x,y) = (0,0), (0,1), (1,0)$$

and at its center,

$$(x,y) = \left(\frac{1}{3}, \frac{1}{3}\right).$$

Evaluating the Jacobian at each of these points and examining its eigenvalues, we find that the three corner points are saddles, with

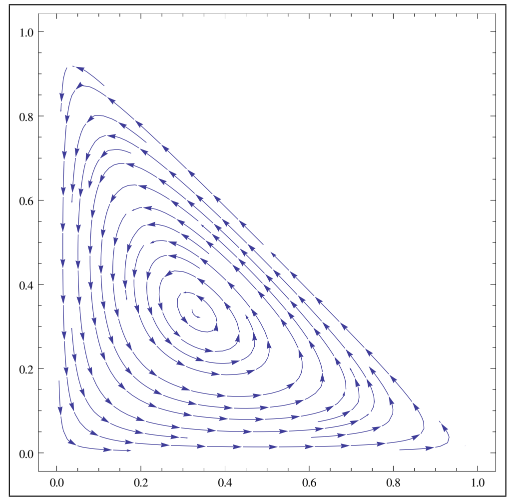

FIGURE 2. Vector field in T for the standard replicator equation $g(x) = x$. The horizontal axis is x and the vertical axis is y.

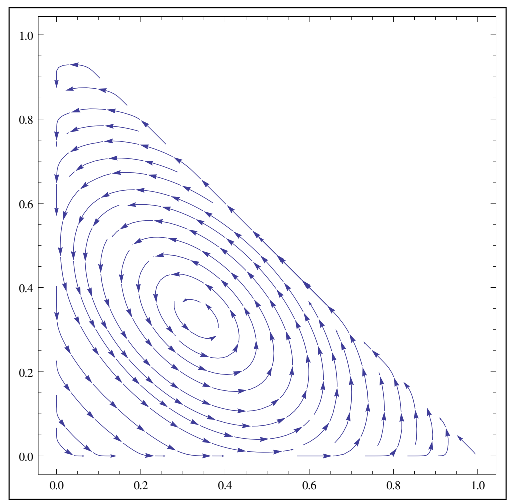

FIGURE 3. Vector field in T for $g(x) = x/(1+ax)$ with $a = 100$. The axes are as above.

$\lambda_{1,2} = \pm 1$. The point $\left(\frac{1}{3}, \frac{1}{3}\right)$ is a linear center, with $\lambda_{1,2} = \pm \frac{i\sqrt{3}}{a+3}$. See Figs. 2 and 3.

As in the case of the standard replicator equation [3], when $g(x) = x/(1+ax)$, the linear center is surrounded by closed periodic orbits. (Although this claim is presented without proof, we will prove it in the next case, $g(x) = x - ax^2$. The proof in this case is largely identical.)

Taking $g(x_i) = x_i - ax_i^2$

The previous choice of $g(x_i)$ did not generate qualitatively different behavior from the usual replicator dynamics. However, it turns out that new behavior occurs when we use $g(x) = x - ax^2$ obtained by truncating the Maclaurin series

$$\frac{x}{1+ax} = x\sum_{n=0}^{\infty}(-ax)^n \tag{16}$$

after the x^2 term.

Thus we consider the case where the growth function is given by $g(x_i) = x_i - ax_i^2$. This represents the assumption that in the absence of competition, population x_i would experience logistic growth. In this case, Eqns. (11) and (12) become

$$\dot{x} = \frac{x(ax-1)(x+2y-1)(a(1+3y(y-1)+x(3y-1))-1)}{a\left(x^2+y^2+(1-x-y)^2\right)-1} \tag{17}$$

$$\dot{y} = -\frac{y(ay-1)(y+2x-1)(a(1+3x(x-1)+y(3x-1))-1)}{a\left(x^2+y^2+(1-x-y)^2\right)-1} \tag{18}$$

The denominators vanish when

$$x^2+y^2+(1-x-y)^2 = \frac{1}{a}. \tag{19}$$

We reject values of a for which Eqn. (19) holds for any $(x,y) \in T$. This happens for $a \in [1,3]$, so we stipulate that $0 \le a < 1$ or $a > 3$. Geometrically, the vector field in T is undefined for values of a such that the sphere $x^2+y^2+z^2 = \frac{1}{a}$ intersects Σ, Eqn. (10).

This system has 13 equilibrium points:

(i) The corners of T

$$(x,y) = (0,0),(0,1),(1,0)$$

(ii) The center of T

$$(x,y) = \left(\frac{1}{3},\frac{1}{3}\right)$$

(iii) Two points on each of the edges $x=0$, $y=0$ and $z=0$

$$(x,y) = \left(0,\frac{1}{a}\right),\left(0,\frac{a-1}{a}\right)$$
$$(x,y) = \left(\frac{1}{a},0\right),\left(\frac{a-1}{a},0\right)$$
$$(x,y) = \left(\frac{1}{a},\frac{a-1}{a}\right),\left(\frac{a-1}{a},\frac{1}{a}\right)$$

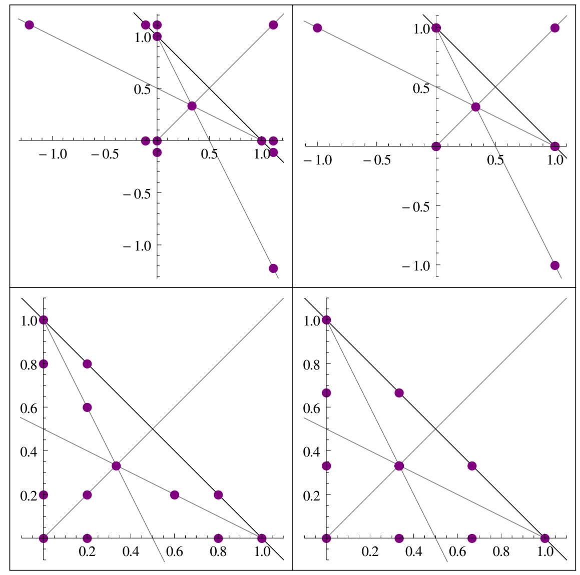

FIGURE 4. Location in the (x,y) plane of the 13 equilibria for $g(x) = x - ax^2$, taking (clockwise from upper left) $a = 0.9$, $a = 1$, $a = 3$, and $a = 5$. The cases $a = 1$ and $a = 3$ are non-physical. As discussed in the text, for $a > 3$ all 13 equilibrium points lie in the region of interest T.

(iv) Three points on lines that pass through the center of T

$$(x,y) = \left(\frac{1}{a},\frac{1}{a}\right),\left(\frac{1}{a},\frac{a-2}{a}\right),\left(\frac{a-2}{a},\frac{1}{a}\right)$$

Figure 4 shows the location of the equilibria. For $0 \le a < 1$, only the corners and the center point lie in T, Eqn. (13), and the dynamics are qualitatively similar to the Rock-Paper-Scissors game with standard replicator dynamics. Evaluating the Jacobian of $[\dot{x},\dot{y}]$ at each equilibrium and computing the eigenvalues, we find that $\left(\frac{1}{3},\frac{1}{3}\right)$ is a linear center and the corner points are saddles.

When $(x,y) = (\frac{1}{3},\frac{1}{3})$,

$$J = \frac{a-3}{9}\begin{pmatrix}1 & 2\\ -2 & -1\end{pmatrix} \Rightarrow \lambda_{1,2} = \frac{\pm i(a-3)}{3\sqrt{3}} \tag{20}$$

and when $(x,y) = (0,0)$,

$$J = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \Rightarrow \lambda_{1,2} = \pm 1. \tag{21}$$

The stability calculations for the other two corner points are similar. Figure 5 shows the vector field and equilibria for $a = \frac{1}{5}$.

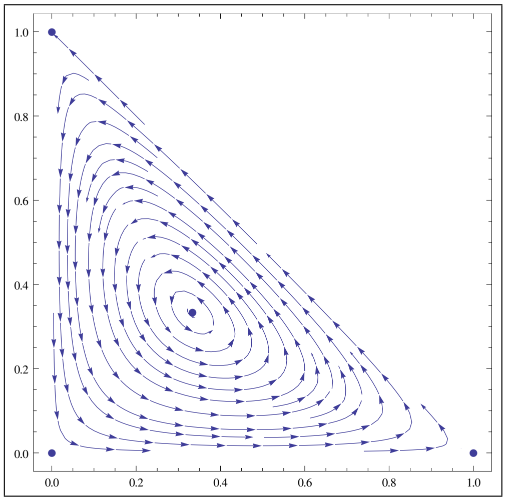

FIGURE 5. **Vector field in T for $a = \frac{1}{5}$.**

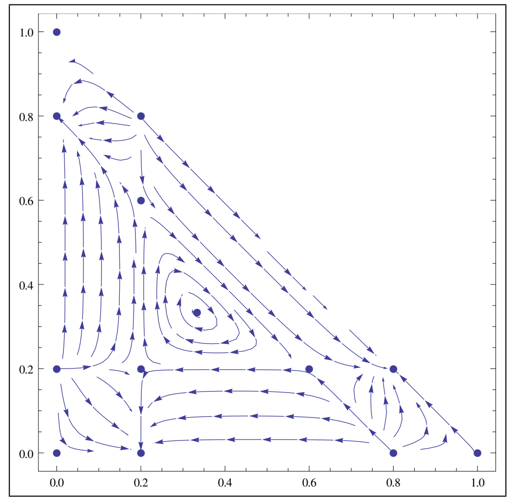

FIGURE 6. **Vector field in T for $a = 5$.**

For $a > 3$, the dynamics are more interesting. All 13 equilibria lie in T. By symmetry, the three equilibria which lie on lines through the center must be of the same type; similarly, the two equilibria on the edge $x = 0$ must be of the same types as their counterparts on the other two edges.

$$(x,y) = \left(\frac{1}{a},\frac{1}{a}\right) \Rightarrow \tag{22}$$

$$J = \begin{pmatrix} \frac{3}{a}-1 & 0 \\ 0 & 1-\frac{a}{3} \end{pmatrix} \Rightarrow \lambda_{1,2} = \pm\left(\frac{3}{a}-1\right)$$

$$(x,y) = \left(0,\frac{1}{a}\right) \Rightarrow \tag{23}$$

$$J = \begin{pmatrix} 1-\frac{3}{a} & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = 1-\frac{3}{a}$$

$$(x,y) = \left(0,1-\frac{1}{a}\right) \Rightarrow \tag{24}$$

$$J = \begin{pmatrix} \frac{3}{a}-1 & 0 \\ -\frac{3}{a} & -1 \end{pmatrix} \Rightarrow \lambda_1 = -1, \lambda_2 = \frac{3}{a}-1.$$

Thus the interior equilibria are saddles, and there is a source and a sink on each edge.

Figures 6 and 7 exhibit another feature of this system: in addition to the boundaries of T, the lines $x = \frac{1}{a}$, $y = \frac{1}{a}$ and $x+y = 1-\frac{1}{a}$ are also invariant, and for $a > 3$, portions of these lines fall within T. Substituting $x = \frac{1}{a}$ into Eqn. (17), we obtain

$$\dot{x} = 0 \tag{25}$$

$$\dot{y} = \tag{26}$$

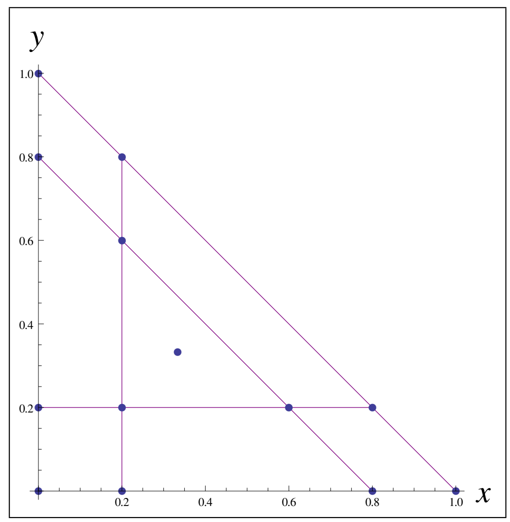

FIGURE 7. **Equilibrium points and invariant lines of the system for $a = 5$.**

$$\frac{\left(y-ay^2\right)\left(y+\frac{2}{a}-1\right)\left(a\left(1+\frac{3}{a}\left(\frac{1}{a}-1\right)+y\left(\frac{3}{a}-1\right)\right)-1\right)}{a\left(\left(\frac{1}{a}\right)^2+y^2+\left(1-\frac{1}{a}-y\right)^2\right)-1}.$$

Similarly, taking $y = \frac{1}{a}$ gives $\dot{y} = 0$. To see that $x+y = 1-\frac{1}{a}$ is an invariant line, we take $y = 1-x-\frac{1}{a}$, so that

$$\dot{x} = \frac{x(a-3)(1+a(x-1))(2+a(x-1))(ax-1)}{a\left(x^2+\left(1-x-\frac{1}{a}\right)^2+\left(\frac{1}{a}\right)^2\right)-1} \tag{27}$$

$$\dot{y} = -\frac{x(a-3)(1+a(x-1))(2+a(x-1))(ax-1)}{a\left(x^2+\left(1-x-\frac{1}{a}\right)^2+\left(\frac{1}{a}\right)^2\right)-1} \quad (28)$$

and $\dot{x}+\dot{y}=0$.

Notice that there appear to be periodic orbits about the equilibrium $\left(\frac{1}{3},\frac{1}{3}\right)$, moving in the opposite direction from before. We will examine this phenomenon more thoroughly in the next section.

FURTHER EXAMINATION OF THE $g(x_i) = x_i - ax_i^2$ CASE

We have observed that the $\left(\frac{1}{3},\frac{1}{3}\right)$ equilibrium is a linear center, and the orbits about it appear to be periodic. To verify this, we show that there is a degenerate Hopf bifurcation in the more general system

$$\dot{x}_i = g(x_i)(f_i - \phi(x)) = g(x_i)\left((A\cdot x)_i - \phi(x)\right) \quad (29)$$

where the payoff matrix is

$$A = \begin{pmatrix} 0 & -a_2 & b_3 \\ b_1 & 0 & -a_1 \\ -a_3 & b_2 & 0 \end{pmatrix} \quad (30)$$

and $\phi(x)$ is defined as before. We substitute this choice of A into Eqn. (29), take the Jacobian, and find that when $(x,y,z) = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ and $a_1 = \cdots = b_3 = 1$, the eigenvalues are

$$\lambda_{1,2} = \pm\frac{i(a-3)}{3\sqrt{3}}, \quad \lambda_3 = 0. \quad (31)$$

Thus there is a Hopf bifurcation at this point in the parameter space, as we might expect from the standard replicator equation [5].

To show that the Hopf bifurcation is in fact degenerate, we follow [6]. First we project the system into the (x,y) plane as before, and make the coordinate translation

$$(x,y) = (u+\frac{1}{3}, v+\frac{1}{3}) \quad (32)$$

to move the bifurcation to the origin. We then write the system as

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = J\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} \quad (33)$$

Then we make a coordinate transformation $u = 2r$, $v = -r - s\sqrt{3}$. This gives the normal form

$$\begin{pmatrix} \dot{r} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} + \begin{pmatrix} h(r,s) \\ k(r,s) \end{pmatrix} \quad (34)$$

where $\omega = (a-3)/3\sqrt{3}$, and h and k are not listed for brevity. Finally, we substitute the resulting nonlinear parts into the equation for the cubic stability coefficient (see [6] pp. 150-155)

$$\begin{aligned} c = {} & \frac{1}{16}[h_{rrr}+h_{rss}+k_{rrs}+k_{sss}] + \\ & \frac{1}{16\omega}[h_{rs}(h_{rr}+h_{ss}) - k_{rs}(k_{rr}+k_{ss}) - h_{rr}k_{rr} + h_{ss}k_{ss}] \end{aligned} \quad (35)$$

and find that $c = 0$. Thus the bifurcation is degenerate.

Generically, as the parameters $a_1,\ldots,b_3$ pass through the critical value $a_1 = \cdots = b_3 = 1$, the equilibrium point at $(x,y) = \left(\frac{1}{3},\frac{1}{3}\right)$ changes from a stable focus to an unstable focus. In what follows we will show that this happens without the appearance of a traditional limit cycle. The family of periodic orbits associated with any Hopf bifurcation will be shown in this case to occur at the critical value, so that the space is filled with closed orbits.

Further symmetries

We have seen that the Hopf bifurcation is degenerate to at least third order. However, it is possible to show by a symmetry argument that the degeneracy extends to all orders, and the orbits inside the region bounded by the invariant lines are periodic.

Note that the flow in Fig. 6 appears conservative in the central region (i.e. all integral curves are closed). However, it is not conservative, as shown by the existence of attracting fixed points. The occurrence of periodic orbits is due to symmetry, not conservative dynamics, as we will now demonstrate.

To show this, we define $(\dot{x},\dot{y},\dot{z})$ using Eqn. (6) with the usual zero-sum RPS payoff matrix and $g(x) = x - ax^2$. We do not eliminate z, but instead define coordinates

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad (36)$$

This is an orthogonal linear transformation such that the plane containing Σ is orthogonal to the w direction, as shown in Figs. 8 and 9. In these coordinates, the point $(x,y,z) = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ is $(u,v,w) = \left(0,0,\frac{1}{\sqrt{3}}\right)$, and $\dot{w} = 0$, so the dynamics can be analyzed in terms of u and v only with no loss of information or symmetry.

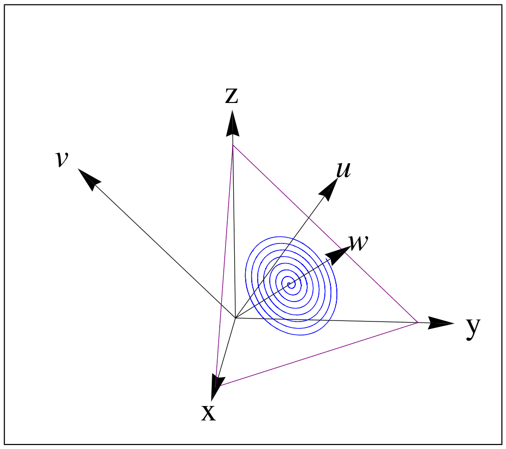

FIGURE 8. **The unit vectors x, y, z, u, v and w shown with a curve in Σ.**

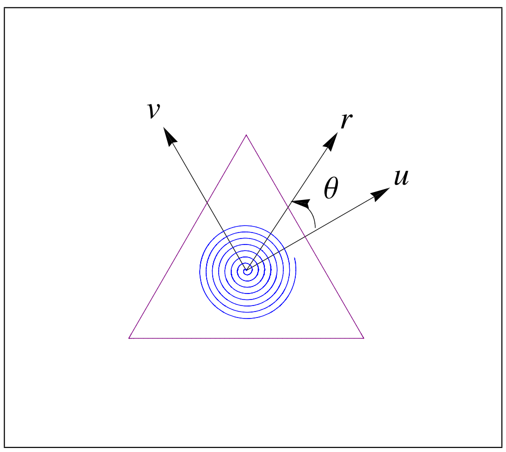

FIGURE 9. **Polar coordinates (r,θ) on Σ. The w direction is out of the page.**

Next, we transform (u,v) into polar coordinates (r,θ) via

$$u = r\cos\theta, \quad v = r\sin\theta. \tag{37}$$

Applying the two successive coordinate changes Eqns. (36) and (37) to Eqn. (29) and solving for $\dot{r}$ and $\dot{\theta}$, we obtain

$$\begin{aligned}\dot{r} &= -\frac{r^2\sin(3\theta)}{6\,(3ar^2+a-3)} \times \\ &\quad \left(\sqrt{2}(a-3)(2a-3)+3\sqrt{3}a^2r^3\cos(3\theta)\right) \\ \dot{\theta} &= -\frac{1}{18\,(3ar^2+a-3)}\left(9\sqrt{3}a^2r^4\cos^2(3\theta)\right. \\ &\quad +\sqrt{3}(a-3)(9ar^2+2a-6) \\ &\quad \left.+3r\sqrt{2}(2a-3)(3ar^2+a-3)\cos(3\theta)\right)\end{aligned} \tag{38, 39}$$

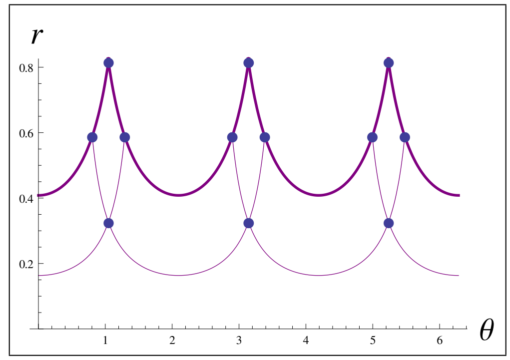

FIGURE 10. **Boundary of Σ, invariant lines, and equilibrium points for $a = 5$, in the (θ,r) plane.**

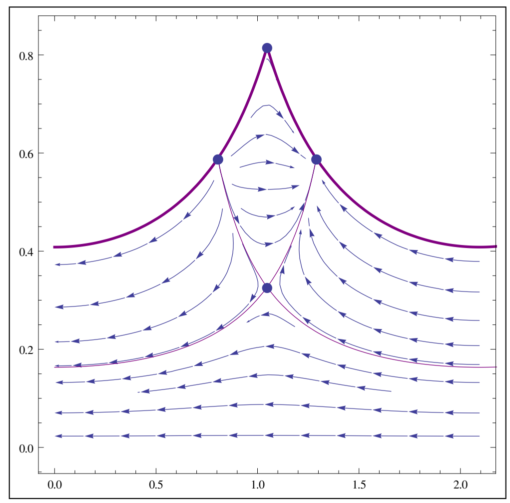

FIGURE 11. **Vector field in (θ,r) plane for $0<\theta<\frac{2\pi}{3}$ and $a=5$. Boundaries of Σ and invariant lines shown.**

Since θ appears only in terms of $\cos(3\theta)$ and $\sin(3\theta)$, we see that the vector field is periodic in θ with period $\frac{2\pi}{3}$. Figures 10 and 11 show the boundaries of Σ and the vector field in the (θ,r) plane.

Finally we show that the central region of Σ is filled with periodic orbits. Notice that $\dot{r}$ is odd, and $\dot{\theta}$ is even, considered as functions of θ. So, if we let

$$\psi = -\theta, \quad \tau = -t \tag{40}$$

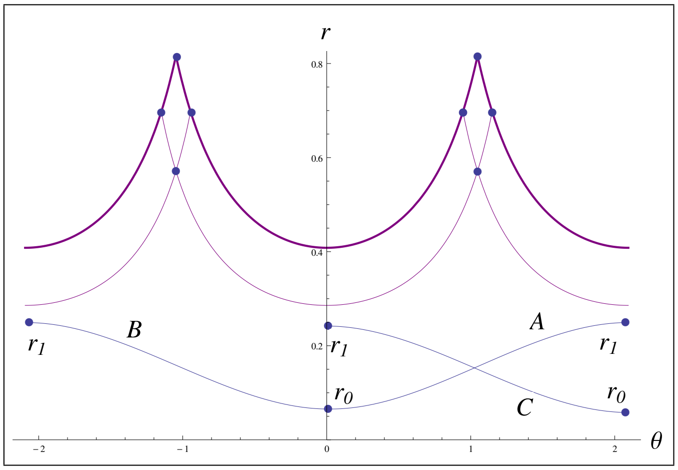

FIGURE 12. **If $r_0 \neq r_1$, then Trajectory A and Trajectory C must cross.**

then

$$\left.\frac{dr}{d\tau}\right|_{(r,\psi)} = \left.\frac{dr}{d(-t)}\right|_{(r,-\theta)} = \left.\frac{dr}{dt}\right|_{(r,\theta)} \tag{41}$$

$$\left.\frac{d\psi}{d\tau}\right|_{(r,\psi)} = \left.\frac{d(-\theta)}{d(-t)}\right|_{(r,-\theta)} = \left.\frac{d\theta}{dt}\right|_{(r,\theta)}. \tag{42}$$

Thus if there is a trajectory (Trajectory A) that starts at $(r,\theta) = (r_0,0)$ at $t=0$ and goes through $(r,\theta) = \left(r_1, \frac{2\pi}{3}\right)$ at $t = t_1$, then there is a matching trajectory (Trajectory B) that starts at $(r,\psi) = (r_0,0)$ at $\tau = 0$ and goes through $(r,\psi) = \left(r_1, \frac{2\pi}{3}\right)$ at $\tau = t_1$.

By the definitions of ψ and τ, Trajectory B in terms of θ starts at $(r,\theta) = \left(r_1, -\frac{2\pi}{3}\right)$ at $t = -t_1$ and goes through $(r,\theta) = (r_0,0)$ at $t=0$. (A schematic of these trajectories is shown in Fig. 12.)

Since $\dot{r}$ and $\dot{\theta}$ are autonomous (hence invariant under translations in time) and $\frac{2\pi}{3}$-periodic in θ, there is a trajectory (Trajectory C) that starts at $(r,\theta) = (r_1,0)$ at $t=0$ and goes through $(r,\theta) = (r_0, \frac{2\pi}{3})$ at $t = t_1$.

In order for this to occur, if $r_0 \neq r_1$, Trajectory A and Trajectory C must cross. This cannot happen since $\dot{r}$ and $\dot{\theta}$ are well-defined functions. Therefore $r_0 = r_1$, and we see that all trajectories that pass through both $\theta = 0$ and $\theta = \frac{2\pi}{3}$ are in fact closed orbits.

CONCLUSION

We have investigated the dynamics of certain systems of the form

$$\dot{x}_i = g(x_i)(f_i - \phi)$$

where $f_i(x) = (A \cdot x)_i$. For $g(x) = x - ax^2$, and the zero-sum RPS choice of A, we find that the system has several fixed points that do not exist in the usual replicator model. It exhibits both periodic motion and convergence to attractors.

This alternate formulation may be instructive in modeling natural or social systems that are not adequately described by the usual replicator dynamics. In particular, this model may apply to systems that exhibit both monotonic and periodic responses depending on initial conditions.

ACKNOWLEDGMENT

Thanks to David Rand for his helpful ideas and intuition when we were beginning work on this project.

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