On Duffing-type oscillators with a straight-line backbone curve

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Summary. This work is concerned with Duffing-type oscillators that have an amplitude-independent frequency/period, i.e. a straight vertical backbone curve. Two groups of oscillators are considered by using a perturbation approach and certain transformation approach: one with a modified Duffing-type restoring force and the other one with the classical hardening and softening Duffing restoring force.

Introduction

Classical hardening and softening Duffing oscillators

$$\ddot{x} + x \pm x^3 = 0, \tag{1}$$

are known to have a frequency ω that changes with their amplitude R [1, 2]. Its backbone curve, which is a graphical representation of the relationship ω=ω(R), is bent either to the right or to the left.

This work aims at answering two questions:

1) Can one modify the model (1) and its restoring force so that the backbone curve unbends to a straight vertical line, corresponding thus to an amplitude-independent frequency/period?

2) Can one design nonlinear oscillators with an amplitude-independent frequency/period without modifying the Duffing restoring force?

Lagrangians and equations of motion

Modified Duffing restoring force

In order to answer the first question we consider the oscillators governed by

$$\ddot{x} + x + a_0 x\dot{x}^2 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots = 0, \tag{2}$$

where a_0 is fixed to 1 or 0, while the coefficients a_2, a_3, a_4 ,… are to be found. These coefficients are calculated by using Lindstedt's method with ω=1, as demonstrated in [3].

In the case when $a_0 = 1$ and there are no even-powered terms in Eq. (2), the coefficients a_{2n+1} ($n = 1,2,3,...$) are found to be $a_{2n+1} = (-1)^n/(2n+1)!!$. Consequently, Eq. (2) can be presented as containing the sum of restoring terms, or in a compact form as follows:

$$\ddot{x} + x\dot{x}^2 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!!} = 0, \quad \text{i.e.} \quad \ddot{x} + x\dot{x}^2 + \sqrt{\frac{\pi}{2}}\exp\left(-\tfrac{x^2}{2}\right)\operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right) = 0, \tag{3a,b}$$

where erfi denotes the imaginary error function. Note that these equations are conservative and Eq. (3b), for example, can be derived by using Lagrange's equation for the Lagrangian $L = \exp\left(x^2\right)\dot{x}^2/2 - \pi\, \operatorname{erfi}^2\left(x/\sqrt{2}\right)/4$. To verify these results, Eq. (3a) was solved numerically and the frequency was extracted from the time response for different values of the initial amplitude R and for various n (note that the initial velocity is assumed to be zero through the whole study). The backbone curves of the corresponding oscillators O_{2n+1} are shown in Figure 1a, where the subscript denotes the highest power included into the sum. The higher the value of n, the more straight the backbone curve is.

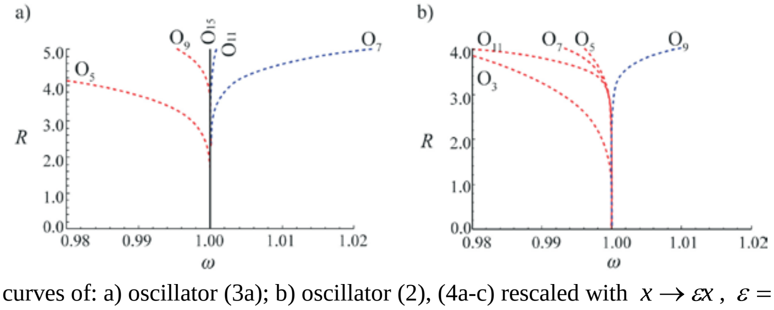

Figure1. Backbone curves of: a) oscillator (3a); b) oscillator (2), (4a-c) rescaled with $x \to \varepsilon x$, $\varepsilon = 0.1$ for $a_0 = 0$

In the case when $a_0 = 0$, the coefficients yielding ω=1 are

$$a_3 = \frac{10}{9}a_2^2 \quad a_5 = \frac{378 a_2 a_4 - 280 a_2^4}{135} \quad a_7 = \frac{48600 a_2 a_6 + 20412 a_4^2 - 186480 a_2^3 a_4 + 148400 a_2^6}{14175}. \tag{4a-c}$$

Figure 1b shows the corresponding backbone curves obtained numerically from Eqs. (2) rescaled with $x \to \varepsilon x$, for the coefficients defined by Eqs. (4a-c) when those of even powers a_2, a_4, $a_{6,\ldots}$ are equal to unity. It is interesting to note that if the coefficients a_2, a_4, $a_{6,}$ are chosen to follow the pattern $a_n = (-1)^{n+1}(2n-1)!!/n!$, Eqs. (4a-c) give the coefficients of odd-powered terms having the values of the same general form. The restoring terms can then be summed up and presented in a compact form so that the equation of motion becomes $\ddot{x} + 1 - 1/\sqrt{1+2x} = 0$, which agrees with Urabe's example of a system exhibiting periodic motion of a fixed period when $-1/2 < x < 1/2$ [4].

Hardening and softening Duffing restoring force

In order to answer the second question from the Introduction, we consider the simple harmonic oscillator with the Lagrangian $L = \dot{X}^2/2 - X^2/2$. By using the transformation $X = x/\sqrt{1 \pm x^2}$ [5, 6], this Lagrangian yields the equation of motion with the Duffing restoring force

$$\ddot{x} \mp \frac{3x}{1 \pm x^2}\dot{x}^2 + x \pm x^3 = 0. \tag{5}$$

For the transformation $X = x \cdot f$, where $f = \exp\left(\int_0^t x(t)dt\right)$ [6], the following equation of motion is derived

$$\ddot{x} + 3x\dot{x} + x + x^3 = 0. \tag{6}$$

The same transformation $X = x \cdot f$ but with $f = \cos\left(\int_0^t x(t)dt\right)$ [6] leads to

$$\ddot{x}_1 - 3\dot{x}_1\ddot{x}_1 \tan x_1 + \dot{x}_1 - \dot{x}_1^3 = 0, \tag{7}$$

where $x_1 = \int_0^t x(t)dt$. Numerical solutions of Eqs. (5)-(7) are plotted for different initial amplitudes in Figures 2a-d as black dots. They confirm that the period is amplitude-independent. This approach also gives analytical solutions for motion [5, 6], which are shown in Figure 2a-d as dashed lines, matching exactly the numerical solutions.

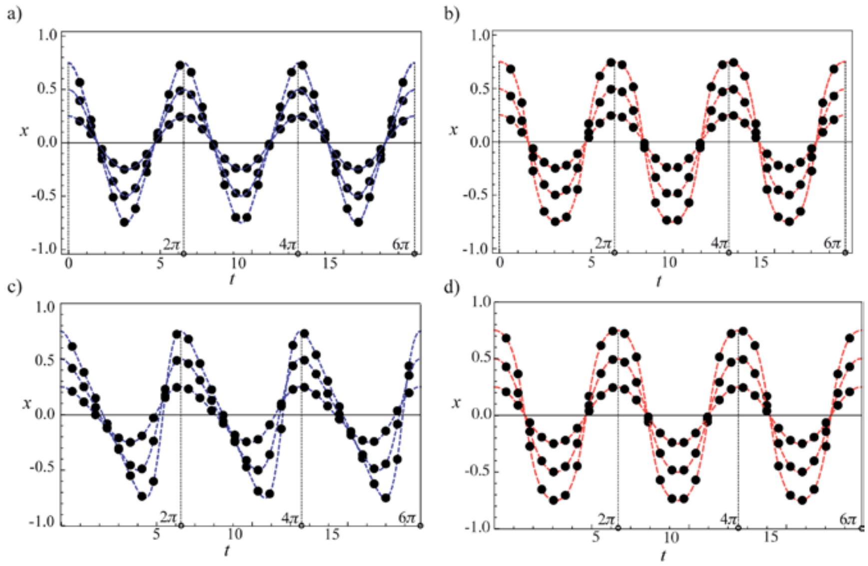

Figure 2. Numerical solutions (dots) and analytical solutions (dashed lines) of: a) Eq. (5) with the upper sign; b) Eq. (5) with the lower sign; c) Eq. (6); d) Eq. (7)

Conclusions

We have designed conservative Duffing-type oscillators whose frequency is amplitude-independent and their backbone curve is consequently vertical and straight. Two approaches are used: a perturbation approach based on Lindstedt's method and the transformation approach that establishes the equivalence between the Lagrangian of the simple harmonic oscillator, which is known to have a constant frequency, and that of the new oscillators. In addition to deriving their Lagrangians and the corresponding equations of motion, analytical solutions for motion and conservation laws are obtained and discussed as well.

References

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