Asymptotic Analysis of the Hopf-Hopf Bifurcation in a Time-delay System

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	Abstract A delay differential equation (DDE) which exhibits a double Hopf or Hopf-Hopf bifurcation [1] is studied using both Lindstedt's method and center manifold reduction. Results are checked by comparison with a numerical continuation program (DDE-BIFTOOL). This work has application to the dynamics of two interacting microbubbles.

1 Introduction

Delay in dynamical systems is exhibited whenever the system's behavior is dependent at least in part on its history. Many technological and biological systems are known to exhibit such behavior; coupled laser systems, high-speed milling, population dynamics and gene expression are some examples of delayed systems. This work analyzes a simple differential delay equation that is motivated by a system of two microbubbles coupled by acoustic forcing, previously studied by Heckman *et al.* [2–5]. The propagation time of sound in the fluid gives rise to a time delay between the two bubbles. The system under study has the same linearization as the equations previously studied, and like them it supports a double Hopf or Hopf-Hopf bifurcation [1] for special values of the system parameters. In order to study the dynamics associated with this type of bifurcation, we replace the nonlinear terms in the original microbubble model with a simpler nonlinearity, namely a cubic term:

$$\kappa \ddot{x} + 4\dot{x} + 4\kappa x + 10\dot{x}_d = \varepsilon x^3. \tag{1}$$

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where $x_d = x(t - T)$.

The case of a typical Hopf bifurcation (not a double Hopf) in a system of DDEs has been shown to be treatable by both Lindstedt's method and center manifold analysis [6,7]. The present paper investigates the use of these methods on a DDE which exhibits a double Hopf. This type of bifurcation occurs when two pairs of complex conjugate roots of the characteristic equation simultaneously cross the imaginary axis in the complex plane. These considerations are dependent only on the linear part of the DDE. If nonlinear terms are present, multiple periodic limit cycles may occur, and in addition to these, quasiperiodic motions may occur, where the quasiperiodicity is due to the two frequencies associated with the pair of imaginary roots in the double Hopf.

Other researchers have investigated Hopf-Hopf bifurcations, as follows. Xu *et al.* [8] developed a method called the perturbation-incremental scheme (PIS) and used it to study bifurcation sets in (among other systems) the van der Pol-Duffing oscillator. They show a robust method for approximating complex behavior both quantitatively and qualitatively in the presence of strong nonlinearities. A similar oscillator system was also studied by Ma *et al.* [9], who applied a center manifold reduction and found quasiperiodic solutions born out of a Neimark-Sacker bifurcation. Such quasiperiodicity in differential-delay equations is well established and has also been studied by e.g. Yu *et al.* [10,11] by investigating Poincaré maps. They also show that chaos naturally evolves via the breakup of tori in the phase space. A study of a general differential delay equation near a nonresonant Hopf-Hopf bifurcation was conducted by Buono *et al.* [12], who also gave a description of the dynamics of a differential delay equation by means of ordinary differential equations on center manifolds.

In this work we analyze a model problem using both Lindstedt's method and center manifold reduction, and we compare results with those obtained by numerical methods, i.e. continuation software.

2 Lindstedt's Method

A Hopf-Hopf bifurcation is characterized by a pair of characteristic roots crossing the imaginary axis at the same parameter value. In order to obtain approximations for the resulting limit cycles, we will first use a version of Lindstedt's method which perturbs off of simple harmonic oscillators. Then the unperturbed solution will have the form:

$$x_0 = A\cos\tau + B\sin\tau$$

where

$$\tau = \omega_i t, \qquad i = 1, 2$$

where $i\omega_1$ and $i\omega_2$ are the associated imaginary characteristic roots.

The example system under analysis is motivated by the Rayleigh-Plesset Equation with Delay Coupling (RPE), as studied by Heckman *et al.* [2,3]. The equation of motion for a spherical bubble contains quadratic nonlinearities and multiple parameters quantifying the fluids' mechanical properties; Equation (1), the object of study in this work, is designed to capture the salient dynamical properties of the RPE while simplifying analysis.

Equation (1) has the same linearization as the RPE, with a cubic nonlinear term added to it. This system has an equilibrium point at $x = 0$ that will correspond to the local behavior of the RPE's equilibrium point as a result.

For $\varepsilon = 0$, equation (1) exhibits a Hopf-Hopf bifurcation with approximate parameters [4]:

$$\kappa = 6.8916, \quad T = T^* = 2.9811,$$
$$\omega_1 = \omega_a = 1.4427, \quad \omega_2 = \omega_b = 2.7726$$

where ω_a, ω_b are values of ω_i at the Hopf-Hopf. As usual in Lindstedt's method we replace t by τ as independent variable, giving

$$\kappa \omega^2 x'' + 4\omega x' + 4\kappa x + 10\omega x_d' = \varepsilon x^3$$

where ω stands for either ω_1 or ω_2. Next we expand x in a power series in ε:

$$x = x_0 + \varepsilon x_1 + \cdots$$

and we also expand ω:

$$\omega = \omega^* + \varepsilon p + \cdots$$

We expect the amplitude of oscillation and the frequency shift p to change in response to a detuning of delay T off of the Hopf-Hopf value T^*:

$$T = T^* + \varepsilon \Delta$$

For the delay term we have:

$$x_d = x_{0_d} + \varepsilon x_{1_d} + \cdots$$

where

$$\begin{aligned} x_{0_d}(\tau) &= x_0(\tau - \omega T) \\ &= x_0(\tau - (\omega^* + \varepsilon p)(T^* + \varepsilon \Delta)) \\ &= x_0(\tau - \omega^* T^*) - \varepsilon(pT^* + \omega^* \Delta)x_0'(\tau - \omega^* T^*) + \cdots \end{aligned}$$

Differentiating,

$$x_d' = x_0'(\tau - \omega^* T^*) - \varepsilon(pT^* + \omega^* \Delta)x_0''(\tau - \omega^* T^*) + \varepsilon x_1'(\tau - \omega^* T^*) + \cdots$$

We introduce the following abbreviation for a delay argument:

$$f(*) = f(\tau - \omega^* T^*)$$

Then

$$x_d' = x_0'(*) - \varepsilon(pT^* + \omega^* \Delta)x_0''(*) + \varepsilon x_1'(*) + \cdots$$

Next we substitute the foregoing expressions into the Eq. (1) which gives

$$\begin{aligned} &\kappa(\omega^* + \varepsilon p)^2(x_0'' + \varepsilon x_1'') + 4(\omega^* + \varepsilon p)(x_0' + \varepsilon x_1') + 4\kappa(x_0 + \varepsilon x_1) \\ &+ 10(\omega^* + \varepsilon p)(x_0'(*) - \varepsilon(pT^* + \omega^* \Delta)x_0''(*) + \varepsilon x_1'(*)) = \varepsilon x_0^3 \end{aligned}$$

and collect terms, giving:

$$\varepsilon^0 : \quad Lx_0 = 0$$

where

$$Lf(\tau) = \kappa\omega^{*2}f'' + 4\omega^*f' + 4\kappa f + 10\omega^*f'(*)$$

$$\varepsilon^1: \quad Lx_1 = -G(x_0)$$

and

$$G(x_0) = 2\kappa\omega^*px_0'' + 4px_0' + 10px_0'(*) - 10\omega^*(pT^* + \omega^*\Delta)x_0''(*) - x_0^3$$

Next we solve $Lx_0 = 0$ for the delayed quantity $x_0'(*)$ with the idea of replacing it in G by non-delayed quantities. We find

$$x_0'(*) = -\frac{1}{10\omega^*}\left\{\kappa\omega^{*2}x_0'' + 4\omega^*x_0' + 4\kappa x_0\right\}$$

Since G contains the quantity $x_0''(*)$, we differentiate the foregoing formula to obtain:

$$x_0''(*) = -\frac{1}{10\omega^*}\left\{\kappa\omega^{*2}x_0''' + 4\omega^*x_0'' + 4\kappa x_0'\right\}$$

We obtain

$$\begin{aligned}
G(x_0) &= 2\kappa\omega^*px_0'' + 4px_0' - \frac{p}{\omega^*}(\kappa\omega^{*2}x_0'' + 4\omega^*x_0' + 4\kappa x_0) \\
&\quad + (pT^* + \omega^*\Delta)(\kappa\omega^{*2}x_0''' + 4\omega^*x_0'' + 4\kappa x_0') - x_0^3 \\
&= \kappa\omega^*px_0'' - \frac{4\kappa p}{\omega^*}x_0 + (pT^* + \omega^*\Delta)(\kappa\omega^{*2}x_0''' + 4\omega^*x_0'' \\
&\quad + 4\kappa x_0') - x_0^3
\end{aligned}$$

Now we take $x_0 = A\cos\tau$ and require the coefficients of $\cos\tau$ and $\sin\tau$ in G to vanish for no secular terms. We obtain:

$$\cos\tau: \quad A\left(-\kappa\omega^*p - \frac{4\kappa p}{\omega^*} + (pT^* + \omega^*\Delta)(-4\omega^*) - \frac{3}{4}A^2\right) = 0$$

$$\sin\tau: \quad A(pT^* + \omega^*\Delta)(\kappa\omega^{*2} - 4\kappa) = 0$$

The second of these gives

$$p = -\frac{\omega^*}{T^*}\Delta$$

whereupon the first gives

$$A^2 = -\frac{4}{3\omega^*}\kappa p(\omega^{*2} + 4)$$

which may be rewritten using the foregoing expression for p:

$$A^2 = \frac{4}{3T^*}\kappa(\omega^{*2} + 4)\Delta$$

Using $\kappa = 6.8916$ and $T^* = 2.9811$, we obtain:

$$A^2 = 3.0823(\omega^{*2} + 4)\Delta$$

This gives

$$A = 4.3295\sqrt{\Delta} \quad \text{for} \quad \omega_a = 1.4427$$

and

$$A = 6.0020\sqrt{\Delta} \quad \text{for} \quad \omega_b = 2.7726$$

3 Center Manifold Reduction

We now approach the same problem via a center manifold reduction method, wherein the critical dynamics of Eq. (1) at the Hopf-Hopf bifurcation are analyzed by seeking a four-dimensional center manifold description corresponding to the codimension-2 Hopf-Hopf.

In order to put Eq. (1) into a form amenable to treatment by functional analysis, we draw on the method used by Kalmár-Nagy *et al.* [13] and Rand [6,7]. The operator differential equation for this system will now be developed. Equation (1) may be written in the form:

$$\dot{\mathbf{x}}(t) = \mathbf{L}(\kappa)\mathbf{x}(t) + \mathbf{R}(\kappa)\mathbf{x}(t-\tau) + \mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-\tau), \kappa)$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\mathbf{L}(\kappa) = \begin{pmatrix} 0 & 1 \\ -4 & -4/\kappa \end{pmatrix}, \ \mathbf{R}(\kappa) = \begin{pmatrix} 0 & 0 \\ 0 & -10/\kappa \end{pmatrix}$$

and

$$\mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-\tau), \kappa) = \begin{pmatrix} 0 \\ (\varepsilon/\kappa)x_1^3 \end{pmatrix}$$

Note that the initial conditions to a differential delay equation consists of a function defined on $-\tau \leq t \leq 0$. As t increases from zero, the initial function on $[-\tau, 0]$ evolves to one on $[-\tau+t, t]$. This implies the flow is determined by a function whose initial conditions are shifting. In order to make the differential delay equation problem tenable to analysis, it is advantageous to recast it in the context of functional analysis.

To accomplish this, we consider a function space of continuously differential functions on $[-\tau, 0]$. The time variable t specifies which function is being considered, namely the one corresponding to the interval $[-\tau+t, t]$. The *phase* variable θ specifies a point in the interval $[-\tau, 0]$.

Now, the variable $\mathbf{x}(t+\theta)$ represents the point in the function space which has evolved from the initial condition function $\mathbf{x}(\theta)$ at time t. From the point of view of the function space, t is now a parameter, whereas θ is the independent variable. To emphasize this new definition, we write

$$\mathbf{x}_t(\theta) = \mathbf{x}(t+\theta), \quad \theta \in [-\tau, 0].$$

The delay differential equation may therefore be expressed as

$$\dot{\mathbf{x}}_t = \mathscr{A}\mathbf{x}_t + \mathscr{F}(\mathbf{x}_t), \tag{2}$$

If κ^* is the critical value of the bifurcation parameter, and noting that $\frac{\partial \mathbf{x}_t(\theta)}{\partial t} = \frac{\partial \mathbf{x}_t(\theta)}{\partial \theta}$ (which follows from $\mathbf{x}_t(\theta) = \mathbf{x}(t+\theta)$), then when $\kappa = \kappa^*$ the operator differential equation has components

$$\mathscr{A}\mathbf{u}(\theta) = \begin{cases} \frac{d}{d\theta}\mathbf{u}(\theta), & \theta \in [-\tau, 0), \\ \mathbf{L}\mathbf{u}(0) + \mathbf{R}\mathbf{u}(-\tau), & \theta = 0, \end{cases} \tag{3}$$

and

$$\mathscr{F}(\mathbf{u}(\theta)) = \begin{cases} \mathbf{0}, & \theta \in [-\tau, 0), \\ \begin{pmatrix} 0 \\ (\varepsilon/\kappa)u_1(0)^3 \end{pmatrix}, & \theta = 0. \end{cases} \tag{4}$$

The linear mapping of the original equation is given by

$$\mathscr{L}(\phi(\theta)) = \mathbf{L}(\kappa)\phi(0) + \mathbf{R}(\kappa)\phi(-\tau)$$

where $\mathbf{x}(t) = \phi(t)$ for $t \in [-\tau, 0]$, $\mathbf{F} : \mathscr{H} \to \mathbb{R}^2$ is a nonlinear functional defined by

$$\mathbf{F}(\phi(\theta)) = \mathbf{f}(\phi(0), \phi(-\tau)),$$

and where $\mathscr{H} = C([-\tau, 0], \mathbb{R}^2)$ is the Banach space of continuously differentiable functions $\mathbf{u} = (u_1, u_2)^T$ from $[-\tau, 0]$ into $\mathbb{R}^2$.

Equations (3) and (4) are representations of Eq. (1) in "canonical form." They contain the corresponding linear and nonlinear parts of Eq. (1) as the boundary conditions to the full evolution equation (2).

A stability analysis of Eq. (3) alone provides insight into the asymptotic stability of the original equations. In the case when Eq. (3) has neutral stability (i.e. has eigenvalues with real part zero), analysis of Eq. (4) is necessary. The purpose of the center manifold reduction is to project the dynamics of the infinite-dimensional singular case onto a low-dimensional subspace on which the dynamics are more analytically tractable.

At a bifurcation, the critical eigenvalues of the operator $\mathscr{A}$ coincide with the critical roots of the characteristic equation. In this system, the target of analysis is a Hopf-Hopf bifurcation, a codimension-2 bifurcation that has a four-dimensional center manifold [14]. Consequently, there will be two pairs of critical eigenvalues $\pm i\omega_a$ and $\pm i\omega_b$ with real part zero. Each eigenvalue has an eigenspace spanned by the real and imaginary parts of its corresponding complex eigenfunction. These eigenfunctions are denoted $\mathbf{s}_a(\theta), \mathbf{s}_b(\theta) \in \mathscr{H}$.

The eigenfunctions satisfy

$$\mathscr{A}\mathbf{s}_a(\theta) = i\omega_a \mathbf{s}_a(\theta),$$
$$\mathscr{A}\mathbf{s}_b(\theta) = i\omega_b \mathbf{s}_b(\theta);$$

or equivalently,

$$\mathscr{A}(\mathbf{s}_{a1}(\theta) + i\mathbf{s}_{a2}(\theta)) = i\omega_a(\mathbf{s}_{a1}(\theta) + i\mathbf{s}_{a2}(\theta)), \tag{5}$$
$$\mathscr{A}(\mathbf{s}_{b1}(\theta) + i\mathbf{s}_{b2}(\theta)) = i\omega_b(\mathbf{s}_{b1}(\theta) + i\mathbf{s}_{b2}(\theta)). \tag{6}$$

Equating real and imaginary parts in Eqs. (5) and (6) gives

$$\mathscr{A}\mathbf{s}_{a1}(\theta) = -\omega_a \mathbf{s}_{a2}(\theta), \tag{7}$$
$$\mathscr{A}\mathbf{s}_{a2}(\theta) = \omega_a \mathbf{s}_{a1}(\theta), \tag{8}$$
$$\mathscr{A}\mathbf{s}_{b1}(\theta) = -\omega_b \mathbf{s}_{b2}(\theta), \tag{9}$$
$$\mathscr{A}\mathbf{s}_{b2}(\theta) = \omega_b \mathbf{s}_{b1}(\theta). \tag{10}$$

Applying the definition of $\mathscr{A}$ to Eqs. (7)–(10) produces the differential equations

$$\frac{d}{d\theta}\mathbf{s}_{a1}(\theta) = -\omega_a \mathbf{s}_{a2}(\theta), \tag{11}$$
$$\frac{d}{d\theta}\mathbf{s}_{a2}(\theta) = \omega_a \mathbf{s}_{a1}(\theta), \tag{12}$$
$$\frac{d}{d\theta}\mathbf{s}_{b1}(\theta) = -\omega_b \mathbf{s}_{b2}(\theta), \tag{13}$$
$$\frac{d}{d\theta}\mathbf{s}_{b1}(\theta) = \omega_b \mathbf{s}_{b1}(\theta), \tag{14}$$

with boundary conditions

$$\mathbf{L}\mathbf{s}_{a1}(0) + \mathbf{R}\mathbf{s}_{a1}(-\tau) = -\omega_a\mathbf{s}_{a2}(0), \tag{15}$$

$$\mathbf{L}\mathbf{s}_{a2}(0) + \mathbf{R}\mathbf{s}_{a2}(-\tau) = \omega_a\mathbf{s}_{a1}(0), \tag{16}$$

$$\mathbf{L}\mathbf{s}_{b1}(0) + \mathbf{R}\mathbf{s}_{b1}(-\tau) = -\omega_b\mathbf{s}_{b2}(0), \tag{17}$$

$$\mathbf{L}\mathbf{s}_{b2}(0) + \mathbf{R}\mathbf{s}_{b2}(-\tau) = \omega_b\mathbf{s}_{b1}(0). \tag{18}$$

The general solution to the differential equations (11)–(14) is:

$$\mathbf{s}_{a1}(\theta) = \cos(\omega_a\theta)\mathbf{c}_{a1} - \sin(\omega_a\theta)\mathbf{c}_{a2},$$
$$\mathbf{s}_{a2}(\theta) = \sin(\omega_a\theta)\mathbf{c}_{a1} + \cos(\omega_a\theta)\mathbf{c}_{a2},$$
$$\mathbf{s}_{b1}(\theta) = \cos(\omega_b\theta)\mathbf{c}_{b1} - \sin(\omega_b\theta)\mathbf{c}_{b2},$$
$$\mathbf{s}_{b2}(\theta) = \sin(\omega_b\theta)\mathbf{c}_{b2} + \cos(\omega_b\theta)\mathbf{c}_{b2},$$

where $\mathbf{c}_{\alpha i} = (c_{\alpha i1}, c_{\alpha i2})^T$. This results in eight unknowns which may be solved by applying the boundary conditions (15)–(18). However, since we are searching for a nontrivial solution to these equations, they must be linearly dependent. We set the value of four of the unknowns to simplify the final result:

$$c_{a11} = 1, \quad c_{a21} = 0, \quad c_{b11} = 1, \quad c_{b21} = 0. \tag{19}$$

This allows Eqs. (15)–(18) to be solved uniquely, yielding

$$\mathbf{c}_{a1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{c}_{a2} = \begin{pmatrix} 0 \\ \omega_a \end{pmatrix}, \quad \mathbf{c}_{b1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{c}_{b2} = \begin{pmatrix} 0 \\ \omega_b \end{pmatrix}.$$

Next, the vectors that span the dual space $\mathscr{H}^*$ must be calculated. The boundary value problem associated with this case has the same differential equations (11)–(14) except on $\mathbf{n}_{\alpha i}$ rather than on $\mathbf{s}_{\alpha i}$; in place of boundary conditions (15)–(18), there are boundary conditions

$$\mathbf{L}^T\mathbf{n}_{a1}(0) + \mathbf{R}^T\mathbf{n}_{a1}(\tau) = \omega_a\mathbf{n}_{a2}(0), \tag{20}$$

$$\mathbf{L}^T\mathbf{n}_{a2}(0) + \mathbf{R}^T\mathbf{n}_{a2}(\tau) = -\omega_a\mathbf{n}_{a1}(0), \tag{21}$$

$$\mathbf{L}^T\mathbf{n}_{b1}(0) + \mathbf{R}^T\mathbf{n}_{b1}(\tau) = \omega_b\mathbf{n}_{b2}(0), \tag{22}$$

$$\mathbf{L}^T\mathbf{n}_{b2}(0) + \mathbf{R}^T\mathbf{s}_{b2}(\tau) = -\omega_b\mathbf{n}_{b1}(0). \tag{23}$$

The general solution to the differential equation associated with this boundary value problem is

$$\mathbf{n}_{a1}(\sigma) = \cos(\omega_a\sigma)\mathbf{d}_{a1} - \sin(\omega_a\sigma)\mathbf{d}_{a2},$$
$$\mathbf{n}_{a2}(\sigma) = \sin(\omega_a\sigma)\mathbf{d}_{a1} + \cos(\omega_a\sigma)\mathbf{d}_{a2},$$
$$\mathbf{n}_{b1}(\sigma) = \cos(\omega_b\sigma)\mathbf{d}_{b1} - \sin(\omega_b\sigma)\mathbf{d}_{b2},$$
$$\mathbf{n}_{b2}(\sigma) = \sin(\omega_b\sigma)\mathbf{d}_{b2} + \cos(\omega_b\sigma)\mathbf{d}_{b2}.$$

Again, these equations are not linearly independent. Four more equations may be generated by orthonormalizing the $\mathbf{n}_{\alpha i}$ and $\mathbf{s}_{\alpha j}$ vectors (conditions on the bilinear form between these vectors):

$$(\mathbf{n}_{a1}, \mathbf{s}_{a1}) = 1, \quad (\mathbf{n}_{a1}, \mathbf{s}_{a2}) = 0, \tag{24}$$

$$(\mathbf{n}_{b1}, \mathbf{s}_{b1}) = 1, \quad (\mathbf{n}_{b1}, \mathbf{s}_{b2}) = 0, \tag{25}$$

where the bilinear form employed is $(\mathbf{v}, \mathbf{u}) = \mathbf{v}^T(0)\mathbf{u}(0) + \int_{-\tau}^{0} \mathbf{v}^T(\xi + \tau)\mathbf{R}\mathbf{u}(\xi)\mathrm{d}\xi$.

Equations (11)–(14) combined with Eqs. (20)–(25) may be solved uniquely for $d_{\alpha ij}$ in terms of the system parameters. Using Eqs. (19) as the values for $\mathbf{c}_{\alpha i}$ and substituting relevant values of the parameters $\kappa^* = 6.8916$, $\tau^* = 2.9811$, $\omega_a = 1.4427$, and $\omega_b = 2.7726$ [4] yields

$$\mathbf{d}_{a1} = \begin{pmatrix} 0.4786 \\ 0.1471 \end{pmatrix}, \quad \mathbf{d}_{a2} = \begin{pmatrix} -0.4079 \\ 0.1726 \end{pmatrix},$$

$$\mathbf{d}_{b1} = \begin{pmatrix} 0.1287 \\ -0.1088 \end{pmatrix}, \quad \mathbf{d}_{b2} = \begin{pmatrix} 0.1570 \\ 0.0892 \end{pmatrix}.$$

4 Flow on the Center Manifold

The solution vector $\mathbf{x}_t(\theta)$ may be understood as follows. The center subspace is four-dimensional and spanned by the vectors $\mathbf{s}_{\alpha i}$. The solution vector is decomposed into four components $y_{\alpha i}$ in the $\mathbf{s}_{\alpha i}$ basis, but it also contains a part that is out of the center subspace. This component is infinite-dimensional, and is captured by the term $\mathbf{w} = (w_1, w_2)^T$ transverse to the center subspace. The solution vector may therefore be written as

$$\mathbf{x}_t(\theta) = y_{a1}(t)\mathbf{s}_{a1}(\theta) + y_{a2}(t)\mathbf{s}_{a2}(\theta) + y_{b1}(t)\mathbf{s}_{b1}(\theta) + y_{b2}(t)\mathbf{s}_{b2}(\theta) + \mathbf{w}(t)(\theta).$$

Note that, by definition,

$$y_{a1}(t) = (\mathbf{n}_{a1}, \mathbf{x}_t)|_{\theta=0}, \tag{26}$$

$$y_{a2}(t) = (\mathbf{n}_{a2}, \mathbf{x}_t)|_{\theta=0}, \tag{27}$$

$$y_{b1}(t) = (\mathbf{n}_{b1}, \mathbf{x}_t)|_{\theta=0}, \tag{28}$$

$$y_{b2}(t) = (\mathbf{n}_{b2}, \mathbf{x}_t)|_{\theta=0}. \tag{29}$$

The nonlinear part of the operator is crucial for transforming the operator differential equation into the canonical form described by Guckenheimer & Holmes. This nonlinear operator is

$$\mathcal{F}(\mathbf{x}_t)(\theta) = \mathcal{F}(y_{a1}(t)\mathbf{s}_{a1} + y_{a2}(t)\mathbf{s}_{a2} + y_{b1}(t)\mathbf{s}_{b1} + y_{b2}(t)\mathbf{s}_{b2} + \mathbf{w}(t))(\theta)$$

$$= \begin{cases} \mathbf{0}, & \theta \in [-\tau, 0), \\ \begin{pmatrix} 0 \\ \frac{\varepsilon}{\kappa}(y_{a1}c_{a11} + y_{a2}c_{a21} + y_{b1}c_{b11} \\ + y_{b2}c_{b21} + w_1(t)(0))^3 \end{pmatrix}, & \theta = 0. \end{cases}$$

In order to derive the canonical form, we take $\frac{d}{dt}$ of $y_{\alpha i}(t)$ from Eqs. (26)–(29) and carry through the differentiation to the factors of the bilinear form. Noting that $\frac{d}{dt}\mathbf{n}_{\alpha i} = 0$,

$$\dot{y}_{\alpha 1} = (\mathbf{n}_{\alpha 1}, \dot{\mathbf{x}}_t)|_{\theta=0} = (\mathbf{n}_{\alpha 1}, \mathscr{A}\mathbf{x}_t + \mathscr{F}(\mathbf{x}_t))|_{\theta=0}$$

$$= (\mathbf{n}_{\alpha 1}, \mathscr{A}\mathbf{x}_t)|_{\theta=0} + (\mathbf{n}_{\alpha 1}, \mathscr{F}(\mathbf{x}_t))|_{\theta=0}$$

$$= (\mathscr{A}^*\mathbf{n}_{\alpha 1}, \mathbf{x}_t)|_{\theta=0} + (\mathbf{n}_{\alpha 1}, \mathscr{F}(\mathbf{x}_t))|_{\theta=0}$$

$$= \omega_\alpha(\mathbf{n}_{\alpha 2}, \mathbf{x}_t)|_{\theta=0} + (\mathbf{n}_{\alpha 1}, \mathscr{F}(\mathbf{x}_t))|_{\theta=0}$$

$$= \omega_\alpha y_{\alpha 2} + \mathbf{n}_{\alpha 1}^T(0)\mathbf{F}$$

and similarly,

$$\dot{y}_{\alpha 2} = -\omega_\alpha y_{\alpha 1} + \mathbf{n}_{\alpha 2}^T(0)\mathbf{F},$$

where $\mathbf{F} = \mathscr{F}(\mathbf{x}_t)(0) = \mathscr{F}(y_{a1}(t)\mathbf{s}_{a1}(0) + y_{a2}(t)\mathbf{s}_{a2}(0) + y_{b1}(t)\mathbf{s}_{b1}(0) + y_{b2}(t)\mathbf{s}_{b2}(0) + \mathbf{w}(t)(0))$, recalling that $\mathscr{F} = \mathscr{F}(\theta)$, and this notation corresponds to setting $\theta = 0$. Substituting in the definition of $\mathbf{n}_{\alpha i}$ and $\mathbf{F}$,

$$\dot{y}_{a1} = \omega_a y_{a2} + \frac{\varepsilon d_{a12}(y_{a1} + y_{b1} + w_1)^3}{\kappa}, \tag{30}$$

$$\dot{y}_{a2} = -\omega_a y_{a1} + \frac{\varepsilon d_{a22}(y_{a1} + y_{b1} + w_1)^3}{\kappa}, \tag{31}$$

$$\dot{y}_{b1} = \omega_b y_{a2} + \frac{\varepsilon d_{b12}(y_{a1} + y_{b1} + w_1)^3}{\kappa}, \tag{32}$$

$$\dot{y}_{b2} = -\omega_b y_{b1} + \frac{\varepsilon d_{b22}(y_{a1} + y_{b1} + w_1)^3}{\kappa}, \tag{33}$$

where we have used Eq. (19). Recall that the center manifold is tangent to the four-dimensional $y_{\alpha i}$ center subspace at the origin and $\mathbf{w}$ may be approximated by a quadratic in $y_{\alpha i}$. Therefore, the terms w_1 in Eqs. (30)–(33) may be neglected, as their contribution is greater than third order, which had previously been neglected. To analyze these Eqs. (30)–(33), a van der Pol transformation is applied:

$$y_{a1}(t) = r_a(t)\cos(\omega_a t + \theta_a(t)),$$
$$y_{a2}(t) = -r_a(t)\sin(\omega_a t + \theta_a(t)),$$
$$y_{b1}(t) = r_b(t)\cos(\omega_b t + \theta_b(t)),$$
$$y_{b2}(t) = -r_b(t)\sin(\omega_b t + \theta_b(t)),$$

which transforms the coupled differential equations (30)–(33) into

$$\dot{r}_a = \frac{\varepsilon}{\kappa}(\cos(t\omega_a + \theta_a)r_a + \cos(t\omega_b + \theta_b)r_b)^3 \tag{34}$$
$$(d_{a12}\cos(t\omega_a + \theta_a) - d_{a22}\sin(t\omega_a + \theta_a)),$$

$$\dot{\theta}_a = \frac{-\varepsilon}{\kappa r_a}(\cos(t\omega_a + \theta_a)r_a + \cos(t\omega_b + \theta_b)r_b)^3 \tag{35}$$
$$(d_{a22}\cos(t\omega_a + \theta_a) + d_{a12}\sin(t\omega_a + \theta_a)),$$

$$\dot{r}_b = \frac{\varepsilon}{\kappa}(\cos(t\omega_a + \theta_a)r_a + \cos(t\omega_b + \theta_b)r_b)^3 \tag{36}$$
$$(d_{b12}\cos(t\omega_b + \theta_b) - d_{b22}\sin(t\omega_b + \theta_b)),$$

$$\dot{\theta}_b = \frac{-\varepsilon}{\kappa r_b}(\cos(t\omega_a + \theta_a)r_a + \cos(t\omega_b + \theta_b)r_b)^3 \tag{37}$$
$$(d_{b22}\cos(t\omega_b + \theta_b) + d_{b12}\sin(t\omega_b + \theta_b)).$$

By averaging the differential equations (35)–(37) over a single period of $t\omega_\alpha + \theta_\alpha$, the θ_α dependence of the $\dot{r}_\alpha$ equations may be eliminated. Note that ω_a and ω_b are non-resonant frequencies, so

averages may be taken independently of one another.

$$\frac{\omega_a}{2\pi} \int_{\theta_a}^{\theta_a + \frac{2\pi}{\omega_a}} \dot{r}_a \mathrm{d}t = \frac{3}{8} \frac{\varepsilon}{\kappa} d_{a12} r_a (2r_b^2 + r_a^2),$$

$$\frac{\omega_b}{2\pi} \int_{\theta_b}^{\theta_b + \frac{2\pi}{\omega_b}} \dot{r}_b \mathrm{d}t = \frac{3}{8} \frac{\varepsilon}{\kappa} d_{b12} r_b (2r_a^2 + r_b^2).$$

According to Guckenheimer & Holmes, the normal form for a Hopf-Hopf bifurcation in polar coordinates is

$$\frac{\mathrm{d}r_a(t)}{\mathrm{d}t} = \mu_a r_a + a_{11} r_a^3 + a_{12} r_a r_b^2 + \mathcal{O}(|r|^5),$$

$$\frac{\mathrm{d}r_b(t)}{\mathrm{d}t} = \mu_b r_b + a_{22} r_b^3 + a_{21} r_b r_a^2 + \mathcal{O}(|r|^5),$$

$$\frac{\mathrm{d}\theta_a(t)}{\mathrm{d}t} = \omega_a + \mathcal{O}(|r|^2),$$

$$\frac{\mathrm{d}\theta_b(t)}{\mathrm{d}t} = \omega_b + \mathcal{O}(|r|^2),$$

where $\mu_i = \Re \frac{\mathrm{d}\lambda_i(\tau^*)}{\mathrm{d}\tau}$, and τ^* is the critical time-delay for the Hopf-Hopf bifurcation (note that this bifurcation is of codimension-2, so both $\tau = \tau^*$ and $\kappa = \kappa^*$ at the bifurcation). Taking the derivative of the characteristic equation with respect to τ and solving for $\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}$ gives

$$\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau} = \frac{5\lambda(\tau)^2}{5 + 2\exp(\tau\lambda(\tau)) - 5\tau\lambda(\tau) + \exp(\tau\lambda(\tau))\kappa\lambda(\tau)}.$$

Letting $\lambda(\tau) = i\omega_\alpha(\tau)$ and substituting in $\tau = \tau^*$, $\kappa = \kappa^*$, as well as ω_a and ω_b respectively yields

$$\mu_a = -0.1500\Delta, \tag{38}$$

$$\mu_b = 0.2133\Delta, \tag{39}$$

where $\Delta = \tau - \tau^*$. This results in the equations for the flow on the center manifold:

$$\dot{r}_a = -0.1500\Delta r_a + 0.0080 r_a (2r_b^2 + r_a^2), \tag{40}$$

$$\dot{r}_b = 0.2133\Delta r_b - 0.0059 r_b (2r_a^2 + r_b^2). \tag{41}$$

To normalize the coefficients and finally obtain the flow on the center manifold in normal form, let $\bar{r}_a = r_a \sqrt{0.0080}$ and $\bar{r}_b = r_b \sqrt{0.0059}$, resulting in:

$$\dot{\bar{r}}_a = -0.1500\Delta \bar{r}_a + \bar{r}_a^3 + 2.7042 \bar{r}_a \bar{r}_b^2,$$

$$\dot{\bar{r}}_b = 0.2133\Delta \bar{r}_b - 1.4792 \bar{r}_a^2 \bar{r}_b - \bar{r}_b^3.$$

This has quantities $a_{11} = 1$, $a_{22} = -1$, $a_{12} = 2.7042$, and $a_{21} = -1.4792$, which implies that this Hopf-Hopf bifurcation has the unfolding illustrated in Fig. 1.

For the calculated a_{ij}, the bifurcation curves in Fig. 1 become A : $\mu_b = -1.4792\mu_a$, B : $\mu_b = -.6992\mu_a$, and C : $\mu_b = -.3697\mu_a$. From Eqs. (38)–(39), system (1) has $\mu_b = -1.422\mu_a$ for the given
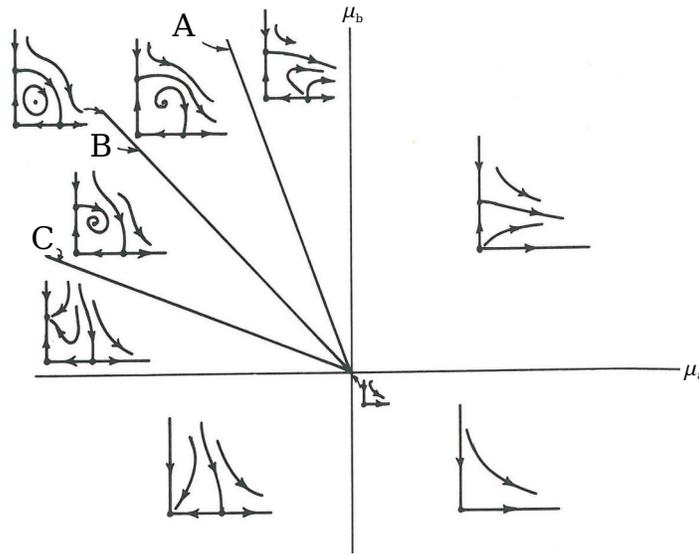
Fig. 1 Partial bifurcation set and phase portraits for the unfolding of this Hopf-Hopf bifurcation. After Guckenheimer & Holmes [14] Figure 7.5.5. Note that the labels A : $\mu_b = a_{21}\mu_a$, B : $\mu_b = \mu_a(a_{21} - 1)/(a_{12} + 1)$, C : $\mu_b = -\mu_a/a_{12}$.

parameter values. Comparison with Fig. 1 shows that this implies the system exhibits two limit cycles with saddle-like stability and an unstable quasiperiodic motion when $\Delta > 0$. We note that the center manifold analysis is local and is expected to be valid only in the neighborhood of the origin.

For comparison, the center manifold reduction Eqs. (40), (41) predict three solutions bifurcating from the Hopf-Hopf (the trivial solution notwithstanding):

$$(r_a, r_b) = \begin{cases} \left(4.3295\sqrt{\Delta}, 0\right), & (42) \\ \left(0, 6.0020\sqrt{\Delta}\right), & (43) \\ \left(4.2148\sqrt{\Delta}, 0.6999\sqrt{\Delta}\right) & \text{(quasiperiodic).} & (44) \end{cases}$$

We note that Eqs. (42), (43) are the same as obtained via Lindstedt's Method in the previous section.

5 Continuation

Figure 2 shows a plot of these results along with those obtained from numerical continuation of the original system (1) with the software package DDE-BIFTOOL [15]. Note that only the two limit cycles are plotted for comparison. The numerical method is seen to agree with the periodic motions predicted by Lindstedt's Method and center manifold reduction.
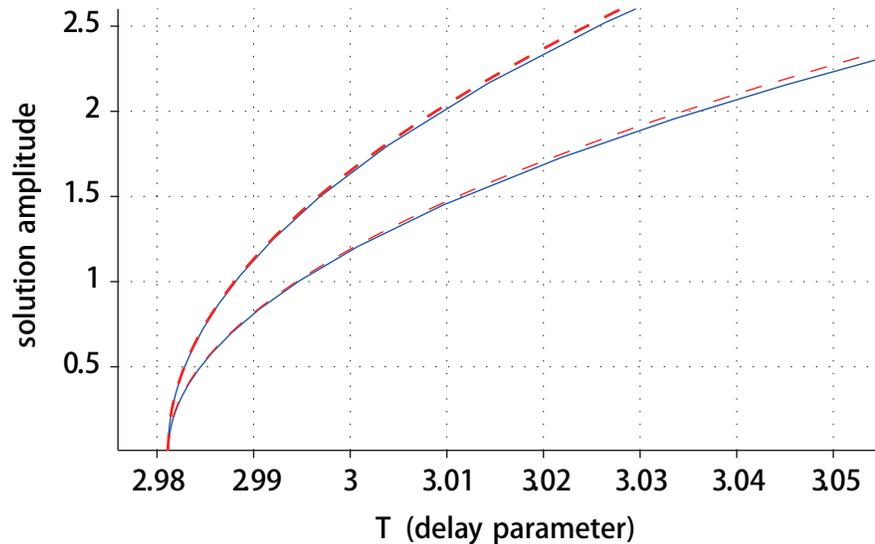
Fig. 2 Comparison of predictions for the amplitudes of limit cycles bifurcating from the Hopf-Hopf point in Eq. (1) obtained by (a) numerical continuation of Eq. (1) using the software DDE-BIFTOOL (solid lines) and (b) center manifold reduction, Equations. (42), (43) (dashed lines).

6 Conclusion

This work has demonstrated agreement between Lindstedt's Method for describing the amplitude growth of limit cycles after a Hopf-Hopf bifurcation and the center manifold reduction of a Hopf-Hopf bifurcation in a nonlinear differential delay equation. While the center manifold reduction analysis is considerably more involved than the application of Lindstedt's Method, it does uncover the quasiperiodic motion which neither Lindstedt's Method nor numerical continuation revealed. Note that in addition to the two limit cycles which were expected to occur due to the Hopf-Hopf bifurcation, the codimension-2 nature of this bifurcation has introduced the possibility of more complicated dynamics than originally anticipated, namely the presence of quasiperiodic motions. This work has served to rigorously show that a system inspired by the physical application of delay-coupled microbubble oscillators exhibits quasiperiodic motions because in part of the occurrence of a Hopf-Hopf bifurcation.

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