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Center Manifold Reduction of the Hopf-Hopf Bifurcation in a Time Delay System

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Résumé. In this work, a differential delay equation with a cubic nonlinearity is analyzed as two parameters are varied by means of a center manifold reduction. This reduction is applied directly to the case where the system undergoes a Hopf-Hopf bifurcation, thereby giving rise to two separate modes of oscillation. In performing the reduction, the system is shown to exhibit quasiperiodic dynamics that are born out of the Hopf-Hopf bifurcation. This system has analogues in coupled microbubble oscillators.

1 Introduction

Delay in dynamical systems is exhibited whenever the system's behavior is dependent at least in part on its history. Many technological and biological systems are known to exhibit such behavior ; coupled laser systems, high-speed milling, population dynamics and gene expression are some examples of delayed systems. This work analyzes a simple differential delay equation that is motivated by a system of two microbubbles coupled by acoustic forcing, previously studied by Heckman *et al.* [1] [2]. The propagation time of sound in the fluid gives rise to a time delay between the two bubbles. The system under study has the same linearization as the equations previously studied, but the sophisticated nonlinear interaction terms in the bubble equations have been replaced by a cubic term in order to provide first insights into the full bubble equations.

In particular, the system of coupled microbubbles has been witnessed to exhibit damped oscillation, excited oscillations (i.e. a stable limit cycle created as a result of a supercritical Hopf bifurcation), and quasiperiodic oscillations. These latter dynamics are unexplained by previous work, but it has been previously suggested that these dynamics are the result of a Neimark-Sacker bifurcation. This work explores this possibility by analyzing the dynamics of an analogous system by means of a center manifold reduction. However, in contrast with previous work, this reduction will analyze the Hopf-Hopf bifurcation that results when two parameters (corresponding to the speed of sound in the fluid and the delay propagation time) are varied.

2 Center Manifold Reduction

The system under analysis is motivated by the Rayleigh-Plesset Equation with Delay Coupling (RPE), as studied by Heckman *et al.* [1] [2]. The equation of motion for a spherical bubble contains quadratic nonlinearities and multiple parameters quantifying the fluids' mechanical properties ; the equations studied in this work are designed to capture salient dynamical properties while simplifying analysis. The system is :

$$\kappa\ddot{x} + 4\dot{x} + 4\kappa x + 10\dot{x}(t-T) = \epsilon x^3. \tag{2.1}$$

Eq. (2.1) has the same linearization as the RPE, with a cubic nonlinear term added to it. This system has an equilibrium point at $x = 0$ that will correspond to the local behavior of the RPE's equilibrium point as a result.

In order to put Eq. (2.1) into a form amenable to treatment by functional analysis, we draw on the method used by Kalmár-Nagy *et al.* [3] and Rand [4]. The operator differential equation for this system will now be developed. Eq. (2.1) may be written in the form :

$$\dot{\mathbf{x}}(t) = \mathbf{L}(\kappa)\mathbf{x}(t) + \mathbf{R}(\kappa)\mathbf{x}(t-\tau) + \mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-\tau), \kappa)$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\mathbf{L}(\kappa) = \begin{pmatrix} 0 & 1 \\ -4 & -4/\kappa \end{pmatrix}, \mathbf{R}(\kappa) = \begin{pmatrix} 0 & 0 \\ 0 & -10/\kappa \end{pmatrix}$$

and

$$\mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-\tau), \kappa) = \begin{pmatrix} 0 \\ (\epsilon/\kappa)x_1^3 \end{pmatrix}$$

Thus, the linear mapping of the original equation is given by

$$\mathcal{L}(\phi(\theta)) = \mathbf{L}(\kappa)\phi(0) + \mathbf{R}(\kappa)\phi(-\tau)$$

and $\mathbf{F} : \mathcal{H} \to \mathbb{R}^2$ is a nonlinear functional defined by

$$\mathbf{F}(\phi(\theta)) = \mathbf{f}(\phi(0), \phi(-\tau))$$

where $\mathcal{H} = C([-\tau, 0], \mathbb{R}^n)$ is the Banach space of continuously differentiable functions $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ from $[-\tau, 0]$ into $\mathbb{R}^n$. The delay differential equation may therefore be expressed as

$$\dot{\mathbf{x}}_t = \mathcal{A}\mathbf{x}_t + \mathcal{F}(\mathbf{x}_t), \tag{2.2}$$

where

$$\mathbf{x}_t(\phi) = \mathbf{x}(t + \phi), \quad \phi \in [-\tau, 0].$$

Note that the subscript t indicates that time is now a parameter to the function whose domain is the delay interval. If κ^* is the critical value of the bifurcation parameter, then when $\kappa = \kappa^*$ the operator differential equation has components

$$\mathcal{A}\mathbf{u}(\theta) = \begin{cases} \frac{d}{d\theta}\mathbf{u}(\theta) & \theta \in [-\tau, 0) \\ \mathbf{Lu}(0) + \mathbf{Ru}(-\tau) & \theta = 0 \end{cases} \tag{2.3}$$

and

$$\mathcal{F}(\mathbf{u}(\theta)) = \begin{cases} \mathbf{0} & \theta \in [-\tau, 0) \\ \begin{pmatrix} 0 \\ (\epsilon/\kappa)\, u_1(0)^3 \end{pmatrix} & \theta = 0 \end{cases} \tag{2.4}$$

Eqs. (2.3) and (2.4) are representations of eq. (2.1) in "canonical form." They contain the corresponding linear and nonlinear parts of eq. (2.1) as the boundary conditions to the full evolution equation (2.2).

A stability analysis of eq. (2.3) alone provides insight into the asymptotic stability of the original equations. In the case when eq. (2.3) has neutral stability (i.e. has eigenvalues with real part zero), analysis of eq. (2.4) is necessary. The purpose of the center manifold reduction is to project the dynamics of the infinite-dimensional singular case onto a low-dimensional subspace on which the dynamics are more analytically tractable.

At a bifurcation, the critical eigenvalues of the operator $\mathcal{A}$ coincide with the critical roots of the characteristic equation. In this system, the target of analysis is a Hopf-Hopf bifurcation, a codimension-2 bifurcation that has a four-dimensional center manifold [5]. Consequently, there will be two pairs of critical eigenvalues $\pm i\omega_a$ and $\pm i\omega_b$ with real part zero. Each eigenvalue has an eigenspace spanned by the real and imaginary parts of its corresponding complex eigenfunction. These eigenfunctions are denoted $\mathbf{s}_a(\theta), \mathbf{s}_b(\theta) \in \mathcal{H}$.

The eigenfunctions satisfy

$$\begin{aligned} \mathcal{A}\mathbf{s}_a(\theta) &= i\omega_a\mathbf{s}_a(\theta) \\ \mathcal{A}\mathbf{s}_b(\theta) &= i\omega_b\mathbf{s}_b(\theta); \end{aligned}$$

or equivalently,

$$\mathcal{A}(\mathbf{s}_{a1}(\theta) + i\mathbf{s}_{a2}(\theta)) = i\omega_a(\mathbf{s}_{a1}(\theta) + i\mathbf{s}_{a2}(\theta)) \tag{2.5}$$

$$\mathcal{A}(\mathbf{s}_{b1}(\theta) + i\mathbf{s}_{b2}(\theta)) = i\omega_b(\mathbf{s}_{b1}(\theta) + i\mathbf{s}_{b2}(\theta)) \tag{2.6}$$

Equating real and imaginary parts in eq. (2.5) and eq. (2.6) gives

$$\mathcal{A}\mathbf{s}_{a1}(\theta) = -\omega_a\mathbf{s}_{a2}(\theta) \tag{2.7}$$

$$\mathcal{A}\mathbf{s}_{a2}(\theta) = \omega_a\mathbf{s}_{a1}(\theta) \tag{2.8}$$

$$\mathcal{A}\mathbf{s}_{b1}(\theta) = -\omega_b\mathbf{s}_{b2}(\theta) \tag{2.9}$$

$$\mathcal{A}\mathbf{s}_{b2}(\theta) = \omega_b\mathbf{s}_{b1}(\theta). \tag{2.10}$$

Applying the definition of $\mathcal{A}$ to eqs. (2.7)- (2.10) produces the differential equations

$$\frac{d}{d\theta}\mathbf{s}_{a1}(\theta) = -\omega_a\mathbf{s}_{a2}(\theta) \tag{2.11}$$
$$\frac{d}{d\theta}\mathbf{s}_{a2}(\theta) = \omega_a\mathbf{s}_{a1}(\theta) \tag{2.12}$$
$$\frac{d}{d\theta}\mathbf{s}_{b1}(\theta) = -\omega_b\mathbf{s}_{b2}(\theta) \tag{2.13}$$
$$\frac{d}{d\theta}\mathbf{s}_{b1}(\theta) = \omega_b\mathbf{s}_{b1}(\theta) \tag{2.14}$$

with boundary conditions

$$\mathbf{L}\mathbf{s}_{a1}(0) + \mathbf{R}\mathbf{s}_{a1}(-\tau) = -\omega_a\mathbf{s}_{a2}(0) \tag{2.15}$$
$$\mathbf{L}\mathbf{s}_{a2}(0) + \mathbf{R}\mathbf{s}_{a2}(-\tau) = \omega_a\mathbf{s}_{a1}(0) \tag{2.16}$$
$$\mathbf{L}\mathbf{s}_{b1}(0) + \mathbf{R}\mathbf{s}_{b1}(-\tau) = -\omega_b\mathbf{s}_{b2}(0) \tag{2.17}$$
$$\mathbf{L}\mathbf{s}_{b2}(0) + \mathbf{R}\mathbf{s}_{b2}(-\tau) = \omega_b\mathbf{s}_{b1}(0) \tag{2.18}$$

The general solution to the differential equations (2.11)-(2.14) is :

$$\begin{aligned}
\mathbf{s}_{a1}(\theta) &= \cos(\omega_a\theta)\mathbf{c}_{a1} - \sin(\omega_a\theta)\mathbf{c}_{a2} \\
\mathbf{s}_{a2}(\theta) &= \sin(\omega_a\theta)\mathbf{c}_{a1} + \cos(\omega_a\theta)\mathbf{c}_{a2} \\
\mathbf{s}_{b1}(\theta) &= \cos(\omega_b\theta)\mathbf{c}_{b1} - \sin(\omega_b\theta)\mathbf{c}_{b2} \\
\mathbf{s}_{b2}(\theta) &= \sin(\omega_b\theta)\mathbf{c}_{b2} + \cos(\omega_b\theta)\mathbf{c}_{b2}
\end{aligned}$$

where $\mathbf{c}_{\alpha i} = \begin{pmatrix} c_{\alpha i1} \\ c_{\alpha i2} \end{pmatrix}$. This results in eight unknowns which may be solved by applying the boundary conditions (2.15)-(2.18). However, since we are searching for a nontrivial solution to these equations, they must be linearly dependent. We set the value of four of the unknowns to simplify the final result :

$$c_{a11} = 1, \quad c_{a21} = 0, \quad c_{b11} = 1, \quad c_{b21} = 0. \tag{2.19}$$

This allows eqs. (2.15)-(2.18) to be solved uniquely, yielding

$$\mathbf{c}_{a1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{c}_{a2} = \begin{pmatrix} 0 \\ \omega_a \end{pmatrix}, \quad \mathbf{c}_{b1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{c}_{b2} = \begin{pmatrix} 0 \\ \omega_b \end{pmatrix}.$$

Next, the vectors that span the dual space $\mathcal{H}^*$ must be calculated. The boundary value problem associated with this case has the same differential equations (2.11)-(2.14) except on $\mathbf{n}_{\alpha i}$ rather than on $\mathbf{s}_{\alpha i}$; in place of boundary conditions (2.15)-(2.18), there are boundary conditions

$$\mathbf{L}^T\mathbf{n}_{a1}(0) + \mathbf{R}^T\mathbf{n}_{a1}(\tau) = \omega_a\mathbf{n}_{a2}(0) \tag{2.20}$$
$$\mathbf{L}^T\mathbf{n}_{a2}(0) + \mathbf{R}^T\mathbf{n}_{a2}(\tau) = -\omega_a\mathbf{n}_{a1}(0) \tag{2.21}$$
$$\mathbf{L}^T\mathbf{n}_{b1}(0) + \mathbf{R}^T\mathbf{n}_{b1}(\tau) = \omega_b\mathbf{n}_{b2}(0) \tag{2.22}$$
$$\mathbf{L}^T\mathbf{n}_{b2}(0) + \mathbf{R}^T\mathbf{s}_{b2}(\tau) = -\omega_b\mathbf{n}_{b1}(0) \tag{2.23}$$

The general solution to the differential equation associated with this boundary value problem is

$$\begin{aligned}
\mathbf{n}_{a1}(\sigma) &= \cos(\omega_a\sigma)\mathbf{d}_{a1} - \sin(\omega_a\sigma)\mathbf{d}_{a2} \\
\mathbf{n}_{a2}(\sigma) &= \sin(\omega_a\sigma)\mathbf{d}_{a1} + \cos(\omega_a\sigma)\mathbf{d}_{a2} \\
\mathbf{n}_{b1}(\sigma) &= \cos(\omega_b\sigma)\mathbf{d}_{b1} - \sin(\omega_b\sigma)\mathbf{d}_{b2} \\
\mathbf{n}_{b2}(\sigma) &= \sin(\omega_b\sigma)\mathbf{d}_{b2} + \cos(\omega_b\sigma)\mathbf{d}_{b2}
\end{aligned}$$

Again, these equations are not linearly independent. Four more equations may be generated by orthonormalizing the $\mathbf{n}_{\alpha i}$ and $\mathbf{s}_{\alpha j}$ vectors (conditions on the bilinear form between these vectors) :

$$(\mathbf{n}_{a1}, \mathbf{s}_{a1}) = 1, \quad (\mathbf{n}_{a1}, \mathbf{s}_{a2}) = 0 \tag{2.24}$$
$$(\mathbf{n}_{b1}, \mathbf{s}_{b1}) = 1, \quad (\mathbf{n}_{b1}, \mathbf{s}_{b2}) = 0 \tag{2.25}$$

where the bilinear form employed is $(\mathbf{v}, \mathbf{u}) = \mathbf{v}^T(0)\mathbf{u}(0) + \int_{-\tau}^{0} \mathbf{v}^T(\xi + \tau)\mathbf{R}\mathbf{u}(\xi)d\xi$.

Eqs. (2.11)-(2.14) combined with (2.20)-(2.25) may be solved uniquely for $d_{\alpha ij}$ in terms of the system parameters. Using eqs. (2.19) as the values for $\mathbf{c}_{\alpha i}$ and substituting relevant values of the parameters $\kappa^* = 6.8916$, $\tau^* = 2.9811$, $\omega_a = 1.4427$, and $\omega_b = 2.7726$ to center the perturbation method at the Hopf-Hopf bifurcation yields

$$\mathbf{d}_{a1} = \begin{pmatrix} 0.4786 \\ 0.1471 \end{pmatrix}, \quad \mathbf{d}_{a2} = \begin{pmatrix} -0.4079 \\ 0.1726 \end{pmatrix},$$

$$\mathbf{d}_{b1} = \begin{pmatrix} 0.1287 \\ -0.1088 \end{pmatrix}, \quad \mathbf{d}_{b2} = \begin{pmatrix} 0.1570 \\ 0.0892 \end{pmatrix}$$

3 Flow on the Center Manifold

The solution vector $\mathbf{x}_t(\theta)$ may be understood as follows. The center subspace is four-dimensional and spanned by the vectors $\mathbf{s}_{\alpha i}$. The solution vector is decomposed into four components $y_{\alpha i}$ in the $\mathbf{s}_{\alpha i}$ basis, but it also contains a part that is out of the center subspace. This component is infinite-dimensional, and is captured by the term $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ transverse to the center subspace. The solution vector may therefore be written as

$$\mathbf{x}_t(\theta) = y_{a1}(t)\mathbf{s}_{a1}(\theta) + y_{a2}(t)\mathbf{s}_{a2}(\theta) + y_{b1}(t)\mathbf{s}_{b1}(\theta) + y_{b2}(t)\mathbf{s}_{b2}(\theta) + \mathbf{w}(t)(\theta)$$

Note that, by definition,

$$y_{a1}(t) = (\mathbf{n}_{a1}, \mathbf{x}_t)|_{\theta=0} \tag{3.1}$$
$$y_{a2}(t) = (\mathbf{n}_{a2}, \mathbf{x}_t)|_{\theta=0} \tag{3.2}$$
$$y_{b1}(t) = (\mathbf{n}_{b1}, \mathbf{x}_t)|_{\theta=0} \tag{3.3}$$
$$y_{b2}(t) = (\mathbf{n}_{b2}, \mathbf{x}_t)|_{\theta=0} \tag{3.4}$$

The nonlinear part of the operator is crucial for transforming the operator differential equation into the canonical form described by Guckenheimer & Holmes. This nonlinear operator is

$$\begin{aligned}\mathcal{F}(\mathbf{x}_t)(\theta) &= \mathcal{F}(y_{a1}(t)\mathbf{s}_{a1} + y_{a2}(t)\mathbf{s}_{a2} + y_{b1}(t)\mathbf{s}_{b1} + y_{b2}(t)\mathbf{s}_{b2} + \mathbf{w}(t))(\theta) \\ &= \begin{cases} \mathbf{0} & \theta \in [-\tau, 0) \\ \begin{pmatrix} 0 \\ \frac{\epsilon}{\kappa}(y_{a1}c_{a11} + y_{a2}c_{a21} + y_{b1}c_{b11} + y_{b2}c_{b21} + w_1(t)(0))^3 \end{pmatrix} & \theta = 0 \end{cases}\end{aligned}$$

In order to derive the canonical form, we take $\frac{d}{dt}$ of $y_{\alpha i}(t)$ from eqs. (3.1)-(3.4) and carry through the differentiation to the factors of the bilinear form. Noting that $\frac{d}{dt}\mathbf{n}_{\alpha i} = 0$,

$$\begin{aligned}\dot{y}_{\alpha 1} &= (\mathbf{n}_{\alpha 1}, \dot{\mathbf{x}}_t)|_{\theta=0} = (\mathbf{n}_{\alpha 1}, \mathcal{A}\mathbf{x}_t + \mathcal{F}(\mathbf{x}_t))|_{\theta=0} \\ &= (\mathbf{n}_{\alpha 1}, \mathcal{A}\mathbf{x}_t)|_{\theta=0} + (\mathbf{n}_{\alpha 1}, \mathcal{F}(\mathbf{x}_t))|_{\theta=0} \\ &= (\mathcal{A}^*\mathbf{n}_{\alpha 1}, \mathbf{x}_t)|_{\theta=0} + (\mathbf{n}_{\alpha 1}, \mathcal{F}(\mathbf{x}_t))|_{\theta=0} \\ &= \omega_\alpha(\mathbf{n}_{\alpha 2}, \mathbf{x}_t)|_{\theta=0} + (\mathbf{n}_{\alpha 1}, \mathcal{F}(\mathbf{x}_t))|_{\theta=0} \\ &= \omega_\alpha y_{\alpha 2} + \mathbf{n}_{\alpha 1}^T(0)\mathbf{F}\end{aligned}$$

and similarly,

$$\dot{y}_{\alpha 2} = -\omega_\alpha y_{\alpha 1} + \mathbf{n}_{\alpha 2}^T(0)\mathbf{F}$$

where $\mathbf{F} = \mathcal{F}(\mathbf{x}_t)(0) = \mathcal{F}(y_{a1}(t)\mathbf{s}_{a1}(0) + y_{a2}(t)\mathbf{s}_{a2}(0) + y_{b1}(t)\mathbf{s}_{b1}(0) + y_{b2}(t)\mathbf{s}_{b2}(0) + \mathbf{w}(t)(0))$, recalling that $\mathcal{F} = \mathcal{F}(\theta)$, and this notation corresponds to setting $\theta = 0$. Substituting in the definition of $\mathbf{n}_{\alpha i}$ and $\mathbf{F}$,

$$\dot{y}_{a1} = \omega_a y_{a2} + \frac{\epsilon d_{a12}(y_{a1} + y_{b1} + w_1)^3}{\kappa} \tag{3.5}$$
$$\dot{y}_{a2} = -\omega_a y_{a1} + \frac{\epsilon d_{a22}(y_{a1} + y_{b1} + w_1)^3}{\kappa} \tag{3.6}$$
$$\dot{y}_{b1} = \omega_b y_{a2} + \frac{\epsilon d_{b12}(y_{a1} + y_{b1} + w_1)^3}{\kappa} \tag{3.7}$$
$$\dot{y}_{b2} = -\omega_b y_{b1} + \frac{\epsilon d_{b22}(y_{a1} + y_{b1} + w_1)^3}{\kappa} \tag{3.8}$$

where we have used eq. (2.19). Recall that the center manifold is tangent to the four-dimensional $y_{\alpha i}$ center subspace at the origin and $\mathbf{w}$ may be approximated by a quadratic in $y_{\alpha i}$. Therefore, the terms w_1 in eqs. (3.5)-(3.8) may be neglected, as their contribution is greater than third order, which had previously been neglected. To analyze this eqs. (3.5)-(3.8), a van der Pol transformation is applied :

$$\begin{aligned} y_{a1}(t) &= r_a(t)\cos(\omega_a t + \theta_a(t)) \\ y_{a2}(t) &= -r_a(t)\sin(\omega_a t + \theta_a(t)) \\ y_{b1}(t) &= r_b(t)\cos(\omega_b t + \theta_b(t)) \\ y_{b2}(t) &= -r_b(t)\sin(\omega_b t + \theta_b(t)) \end{aligned}$$

transforms the coupled differential equations (3.5)-(3.8) into

$$\dot{r}_a = \frac{\epsilon}{\kappa}(\cos(t\omega_a + \theta_a)r_a + \cos(t\omega_b + \theta_b)r_b)^3(d_{a12}\cos(t\omega_a + \theta_a) - d_{a22}\sin(t\omega_a + \theta_a)) \tag{3.9}$$

$$\dot{\theta}_a = \frac{-\epsilon}{\kappa r_a}(\cos(t\omega_a + \theta_a)r_a + \cos(t\omega_b + \theta_b)r_b)^3(d_{a22}\cos(t\omega_a + \theta_a) + d_{a12}\sin(t\omega_a + \theta_a)) \tag{3.10}$$

$$\dot{r}_b = \frac{\epsilon}{\kappa}(\cos(t\omega_a + \theta_a)r_a + \cos(t\omega_b + \theta_b)r_b)^3(d_{b12}\cos(t\omega_b + \theta_b) - d_{b22}\sin(t\omega_b + \theta_b)) \tag{3.11}$$

$$\dot{\theta}_b = \frac{-\epsilon}{\kappa r_b}(\cos(t\omega_a + \theta_a)r_a + \cos(t\omega_b + \theta_b)r_b)^3(d_{b22}\cos(t\omega_b + \theta_b) + d_{b12}\sin(t\omega_b + \theta_b)) \tag{3.12}$$

By averaging the differential equations (3.9)-(3.12) over a single period of $t\omega_\alpha + \theta_\alpha$, the θ_α dependence of the $\dot{r}_\alpha$ equations may be eliminated. Note that ω_a and ω_b are non-resonant frequencies, so averages may be taken independently of one another.

$$\begin{aligned} \frac{\omega_a}{2\pi}\int_{\theta_a}^{\theta_a+\frac{2\pi}{\omega_a}} \dot{r}_a \mathrm{d}t &= \frac{3}{8}\frac{\epsilon}{\kappa} d_{a12} r_a (2r_b^2 + r_a^2) \\ \frac{\omega_b}{2\pi}\int_{\theta_b}^{\theta_b+\frac{2\pi}{\omega_b}} \dot{r}_b \mathrm{d}t &= \frac{3}{8}\frac{\epsilon}{\kappa} d_{b12} r_b (2r_a^2 + r_b^2) \end{aligned}$$

According to Guckenheimer & Holmes, the normal form for a Hopf-Hopf bifurcation in polar coordinates is

$$\begin{aligned} \frac{\mathrm{d}r_a(t)}{\mathrm{d}t} &= \mu_a r_a + a_{11}r_a^3 + a_{12}r_a r_b^2 + O(|r|^5) \\ \frac{\mathrm{d}r_b(t)}{\mathrm{d}t} &= \mu_b r_b + a_{22}r_b^3 + a_{21}r_b r_a^2 + O(|r|^5) \\ \frac{\mathrm{d}\theta_a(t)}{\mathrm{d}t} &= \omega_a + O(|r|^2) \\ \frac{\mathrm{d}\theta_b(t)}{\mathrm{d}t} &= \omega_b + O(|r|^2) \end{aligned}$$

where $\mu_i = \Re\frac{\mathrm{d}\lambda_i(\tau^*)}{\mathrm{d}\tau}$, and τ^* is the critical time-delay for the Hopf-Hopf bifurcation (note that this bifurcation is of codimension 2, so both $\tau = \tau^*$ and $\kappa = \kappa^*$ at the bifurcation). Taking the derivative of the characteristic equation with respect to τ and solving for $\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}$ gives

$$\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau} = \frac{5\lambda(\tau)^2}{5 + 2\exp(\tau\lambda(\tau)) - 5\tau\lambda(\tau) + \exp(\tau\lambda(\tau))\kappa\lambda(\tau)}.$$

Letting $\lambda(\tau) = i\omega_\alpha(\tau)$ and substituting in $\tau = \tau^*$, $\kappa = \kappa^*$, as well as ω_a and ω_b respectively yields

$$\mu_a = -0.1500\Delta \tag{3.13}$$

$$\mu_b = 0.2133\Delta \tag{3.14}$$

where $\Delta = \tau - \tau^*$. This results in the equations for the flow on the center manifold :

$$\begin{aligned} \dot{r}_a &= -0.1500\Delta r_a + 0.0080 r_a(2r_b^2 + r_a^2) \\ \dot{r}_b &= 0.2133\Delta r_b - 0.0059 r_b(2r_a^2 + r_b^2) \end{aligned}$$

To normalize the coefficients and finally obtain the flow on the center manifold in normal form, let $\bar{r}_a = r_a\sqrt{0.0080}$ and $\bar{r}_b = r_b\sqrt{0.0059}$, and drop the bars, resulting in :

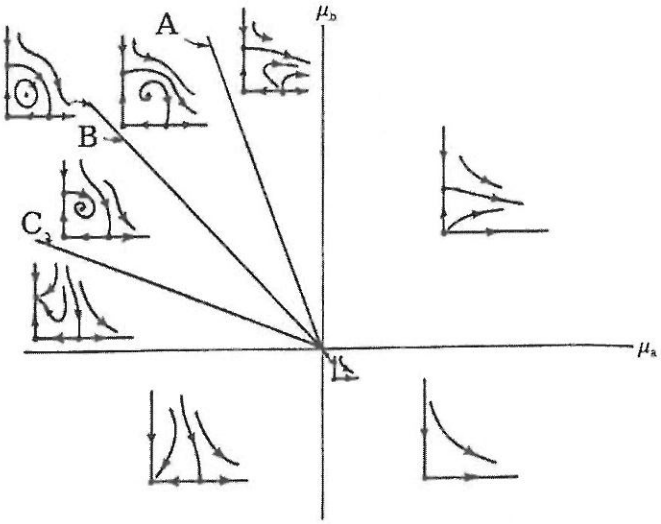

Figure 1. Partial bifurcation set and phase portraits for the unfolding of this Hopf-Hopf bifurcation. Figure reproduced from Guckenheimer & Holmes [5] Figure 7.5.5. Note that the labels A : $\mu_b = a_{21}\mu_a$, B : $\mu_b = \mu_a(a_{21}-1)/(a_{12}+1)$, C : $\mu_b = -\mu_a/a_{12}$.

$$\dot r_a = -0.1500\Delta r_a + r_a^3 + 2.7042 r_a r_b^2$$
$$\dot r_b = 0.2133\Delta r_b - 1.4792 r_a^2 r_b - r_b^3$$

This has quantities $a_{11} = 1$, $a_{22} = -1$, $a_{12} = 2.7042$, and $a_{21} = -1.4792$, which implies that this Hopf-Hopf bifurcation has the unfolding illustrated in Figure 1.

For the calculated a_{ij}, the bifurcation curves in Figure 1 become A : $\mu_b = -1.4792\mu_a$, B : $\mu_b = -.6992\mu_a$, and C : $\mu_b = -.3697\mu_a$. From eqs. (3.13)-(3.14), system (2.1) has $\mu_b = -1.422\mu_a$ for the given parameter values. Comparison with Figure 1 shows that this implies the system exhibits two unstable limit cycles and an unstable quasiperiodic motion when $\Delta > 0$. We note that the center manifold analysis is local and is expected to be valid only in the neighborhood of the origin.

4 Conclusion

This work explored the center manifold reduction of a Hopf-Hopf bifurcation in a nonlinear differential delay equation. When analyzing a system of coupled oscillators that separately undergo Hopf bifurcation, there exists the possibility of the full system to undergo this codimension 2 bifurcation. In doing so, a wealth of sophisticated dynamics may arise that are not immediately anticipated, for instance the quasiperiodic motions. This work has served to rigorously show that a system inspired by the physical application of delay-coupled microbubble oscillators exhibits quasiperiodic motions because in part of the occurrence of a Hopf-Hopf bifurcation.

Références

[1] Heckman C.R., Sah S.M. and Rand R.H., "Dynamics of microbubble oscillators with delay coupling" *Commun Nonlinear Sci Numer Simulat* **15**, (2010) 2735–2743.

[2] Heckman C.R. and Rand R.H., "Dynamics of Coupled Microbubbles with Large Fluid Compressibility Delays," *Proc. EUROMECH 2011 Euro. Nonlin. Osc. Conf.*

[3] Kalmár-Nagy T., Stépán G. and Moon F.C., "Subcritical Hopf Bifurcation in the Delay Equation Model for Machine Tool Vibrations," *Nonlinear Dynamics* **26**, (2001) 121–142.

[4] Rand R.H., *Lecture Notes in Nonlinear Vibrations*, version 52 (2005). `http://www.math.cornell.edu/~rand/randdocs/nlvibe52.pdf`.

[5] Guckenheimer J. and Holmes P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer, 1983)