ON NONLINEAR DIFFERENTIAL EQUATIONS WITH DELAYED SELF-FEEDBACK

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Summary This work concerns the dynamics of nonlinear systems that are subjected to delayed self-feedback. Perturbation methods applied to such systems give rise to slow flows which characteristically contain delayed variables. We consider two approaches to analyzing Hopf bifurcations in such slow flows. In one approach, which we refer to as approach **I**, we follow many researchers in replacing the delayed variables in the slow flow with non-delayed variables, thereby reducing the differential-delay equation (DDE) slow flow to an ordinary differential equation (ODE). In a second approach, which we refer to as approach **II**, we keep the delayed variables in the slow flow. By comparing these two approaches we are able to assess the accuracy of making the simplifying assumption which replaces the DDE slow flow by an ODE.

INTRODUCTION

When investigating a differential-delay equation (DDE) by use of a perturbation method, one is often confronted with a slow flow which contains delay terms. It is usually argued that since the parameter of perturbation, call it ϵ, is small, $\epsilon << 1$, the delay terms which appear in the slow flow may be replaced by the same term without delay, see e.g. [1, 2, 3]. The purpose of the present paper is to analyze the slow flow with the delay terms left in it, and to compare the resulting approximation with the usual one in which the delay terms have been replaced by terms without delay.

EXAMPLE 1: DUFFING EQUATION

For example take the following version of the Duffing equation with delayed self-feedback.

$$\ddot{x} + x = \epsilon \left[-\alpha \dot{x} - \gamma x^3 + k\, x_d \right] \tag{1}$$

where $x_d = x(t-T)$, where $T =$ delay. We treat eq.(1) with the two variable perturbation method, where $x(\xi, \eta)$, where $\xi = t$ and $\eta = \epsilon t$. We expand $x = x_0 + \epsilon x_1 + O(\epsilon^2)$ and obtain the following equation on x_0:

$$Lx_0 \equiv x_{0\xi\xi} + x_0 = 0 \quad \Rightarrow \quad x_0(\xi,\eta) = A(\eta)\cos\xi + B(\eta)\sin\xi \tag{2}$$

Eliminating secular terms in the x_1 equation gives the following slow flow:

$$\frac{dA}{d\eta} = -\alpha\,\frac{A}{2} + \frac{3\,\gamma\,B^3}{8} + \frac{\gamma\,A^2 B}{8} - \frac{k}{2} A_d \sin T - \frac{k}{2} B_d \cos T \tag{3}$$

$$\frac{dB}{d\eta} = -\alpha\,\frac{B}{2} - \frac{3\,\gamma\,A^3}{8} - \frac{\gamma\,A B^2}{8} - \frac{k}{2} B_d \sin T + \frac{k}{2} A_d \cos T \tag{4}$$

where $A_d = A(\eta - \epsilon T)$ and $B_d = B(\eta - \epsilon T)$ are delay terms in the slow flow.

Method I involves replacing the delay terms A_d, B_d in the slow flow (3),(4) respectively by undelayed terms A, B, resulting in a slow flow of ODEs. It is argued that such a step is justified if the product ϵT is small:

$$A_d = A(\eta - \epsilon T) \approx A(\eta) + O(\epsilon), \qquad B_d = B(\eta - \epsilon T) \approx B(\eta) + O(\epsilon). \tag{5}$$

Method II involves studying the slow flow (3),(4) as it is.

Figure 1 (left side) shows a comparison, in the case of the Duffing equation (1), between the analytical Hopf conditions obtained via the two approaches and the numerical Hopf curves. The approach **II** plotted by red/dashed curves gives a better result than the approach **I** (black/dashdot curves). Therefore in the case of Duffing equation, treating the slow flow as a DDE gives better results than approximating the DDE slow flow by an ODE.

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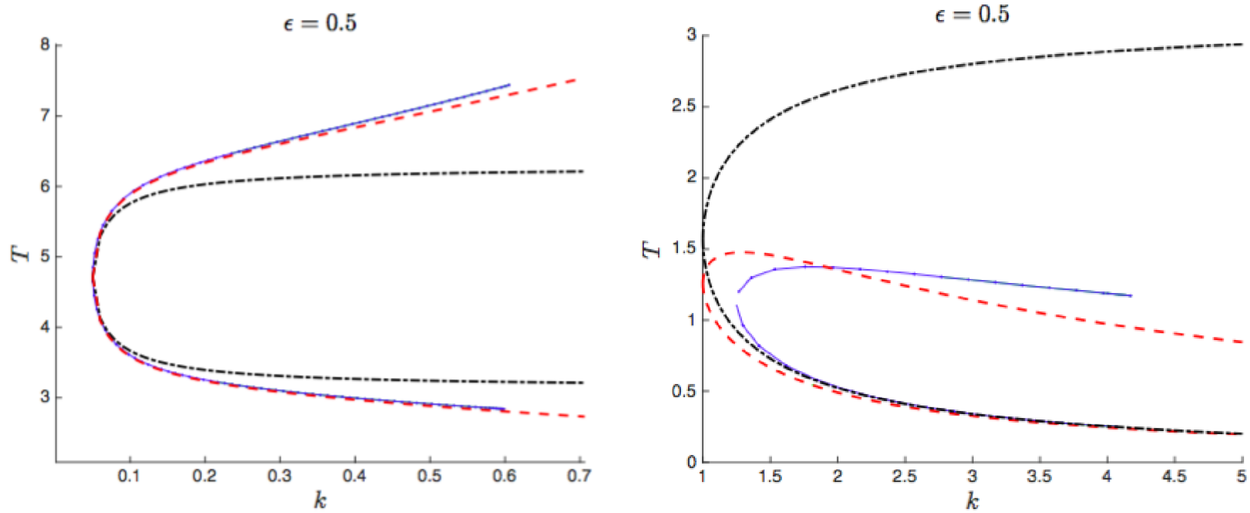

Duffing example, eq.(1) van der Pol example, eq.(6)

Figure 1: Numerical Hopf bifurcation curves (blue/solid). Also shown are the results of approach **I** (black/dashdot), and the results of approach **II** (red/dashed)

EXAMPLE 2: VAN DER POL EQUATION

As a second example, we consider a version of the van der Pol equation with delayed self-feedback [4]:

$$\ddot{x} + x = \epsilon \left[\dot{x}(1 - x^2) + k\, x_d - k\, x\right] \tag{6}$$

We apply the same procedure here as we did for the Duffing equation (1) and show the results in Figure 1 (right). Figure 1 (right) shows that approach **II** gives better results than approach **I**. However approach **I** still gives a good fit for the lower Hopf curve.

CONCLUSIONS

When a DDE with delayed self-feedack is treated using a perturbation method (such as the two variable expansion method, multiple scales, or averaging), the resulting slow flow typically involves delayed variables. In this work we compared the behavior of the resulting DDE slow flow with a related ODE slow flow obtained by replacing the delayed variables in the slow flow with non-delayed variables. We studied sample systems based on the Duffing equation with delayed self-feedback, eq.(1), and on the van der Pol equation with delayed self-feedback, eq.(6). In both cases we found that replacing the delayed variables in the slow flow by non-delayed variables (approach **I**) gave better results on the lower Hopf curve than on the upper Hopf curve.

Our conclusion is therefore that the researcher is advised to perform the more lengthy approach **II** analysis on the DDE slow flow in situations where values of the product ϵT is relatively large, as in the upper Hopf curves in Figures 1.

References

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