12TH CONFERENCE
on
DYNAMICAL SYSTEMS
THEORY AND APPLICATIONS
December 2–5, 2013. Łódź, Poland

Nonlinear oscillators with amplitude-independent frequency

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Abstract: Huygens is believed to have been the first to pose and solve the problem of a nonlinear oscillator that performs "isochronous" oscillations, i.e. oscillations whose frequency and period are amplitude-independent. The present study shows that Huygens' results can be obtained by establishing the equivalence between the kinetic and potential energy of his pendulum and that of a simple harmonic oscillator. Moreover, we are able to generalize this approach to apply to two different types of single-degree-of-freedom nonlinear oscillators whose equations of motion contain either a quadratic or linear term with respect to the generalized velocity. Conditions under which such systems have an isochronous centre at the origin are discussed. General expressions for the corresponding equation of motion, conservation laws as well as solutions for motion and for phase trajectories are also obtained. Several examples are given to illustrate the findings. Numerical simulations are carried out to verify that these nonlinear oscillators have an amplitude-independent period.

1. Introduction

Nonlinear oscillators are in general known to have a frequency/period that depends on their amplitude. Huygens is believed to have been the first to pose and solve the problem of a nonlinear oscillator that performs oscillations whose frequency/period are amplitude-independent [1], [2], which are the so-called isochronous oscillations [3]. It was more than three centuries ago when he showed that if a pendulum wraps around a cycloid, it oscillates isochronously [1], [2].

Recent investigations of isochronicity have mainly been directed towards two classes of nonlinear oscillators, i.e. two types of differential equations of motion: one with a term quadratic in the generalized velocity and the second one with a term linear in the generalized velocity. The former are governed by

$$\ddot{x} + p(x)\dot{x}^2 + q(x) = 0. \tag{1}$$

This equation was studied by Sabatini [4], who derived a sufficient condition for its solution to be oscillatory, i.e. for the origin to be a center: $xq(x) > 0.$ Sabatini also proved that when $p(x)$ and $q(x)$

are odd and analytic, and $xq(x) > 0$ is satisfied for small values of $x \neq 0$, the origin is an isochronous center if and only if the following expression is equal to zero in the whole domain

$$x[q(x)\Phi'(x) - \Phi(x)q'(x) - \Phi(x)q(x)p(x)] \equiv 0, \tag{2}$$

where

$$\Phi(x) = \int_0^x \exp(P(s))ds, \quad P(x) = \int_0^x p(s)ds. \tag{3a,b}$$

Sabatini further gave a characterization of isochronous centres: when $p(x)$ and $q(x)$ are polynomials and the condition $xq(x) > 0$ is satisfied, the origin represents a global isochronous centre if and only if both $p(x)$ and $q(x)$ have an odd degree and $p(x)$ has a positive leading coefficient.

Sabatini [5] also investigated Liénard-type equations, which have a term linear in the generalized velocity

$$\ddot{x} + u(x)\dot{x} + v(x) = 0. \tag{4}$$

He gave necessary and sufficient mathematical conditions for isochronicity in terms of the coefficient functions $u(x)$ and $v(x)$: Let $u(x)$, $v(x)$ be analytic, $v(x)$ odd, $u(0) = v(0) = 0$, $v'(0) > 0$; then the origin O is a center if and only if $u(x)$ is odd, and O is an isochronous center if and only if

$$\left(\int_0^x su(s)ds\right)^2 - x^3(v(x) - v'(0)x) \equiv 0. \tag{5}$$

He illustrated the existence of this behaviour in the system (4) with

$$u(x) = (2n+3)x^{2n+1}, \quad v(x) = x + x^{4n+3}, \tag{6a,b}$$

where n is a non-negative integer. Iacono and Russo showed that this system can be explicitly solved [6]. Necessary and sufficient mathematical conditions for the isochronicity of the differential equation (4) have also been provided by Christopher and Devlin [7]. Chandrasekar et al [8] investigated in detail the so-called modified Emden equation, which is a Liénard-type nonlinear oscillator (4) with

$$u(x) = kx, \quad v(x) = \lambda_1 x + \frac{k^2}{9}x^3, \tag{7a,b}$$

and determined the conditions under which it can yield isochronous oscillations.

The existing theories related to the oscillators modelled by Eqs. (1) and (4) are of a mathematical nature, and neither link the equations of motion with mechanical models, nor all provide general solutions for their isochronous motion. The study proposed in this paper complements previous work by presenting a transformation approach in which the equivalence between the kinetic and potential energy of nonlinear oscillators and that of a simple harmonic oscillator, which is known to perform isochronous oscillations around the origin, is established. Being the motivation for this study, the main results related to Huygens' pendulum and the basics of the transformation approach are presented first in the next section and then, two different classes of single-degree-of-freedom (1DOF) nonlinear isochronous oscillators whose equations of motion have the forms (1) or (4) are studied.

2. On the motivation: Huygens' pendulum

Huygens [1], [2] showed that if a pendulum of length L and mass m wraps around a cycloid

$$x = \frac{L}{4}(\theta - \sin\theta), \quad y = -\frac{L}{4}(\cos\theta - 1), \tag{8a,b}$$

it performs isochronous oscillations. The corresponding parametric equations of motion of the bob are [2]

$$X = \frac{L}{4}(\theta + \sin\theta), \quad Y = -\frac{L}{4}(3 + \cos\theta). \tag{9a,b}$$

Both of these cycloids are shown in Figure 1 for L=4 and for $-\pi \le \theta \le \pi$.

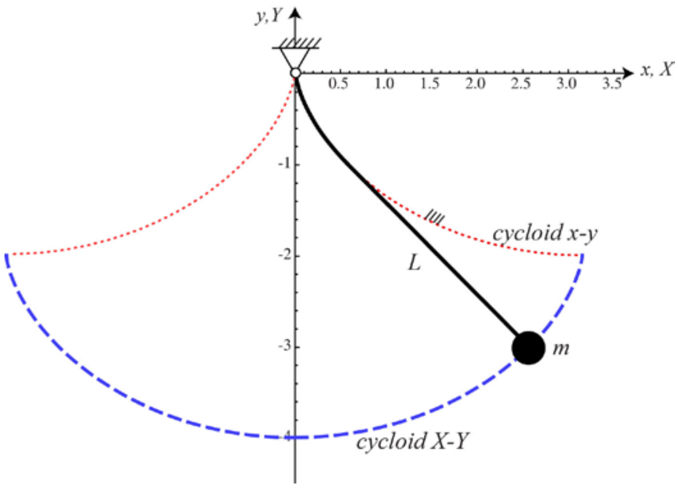

Figure 1. Huygens' pendulum.

By using Eqs. (9a,b), the corresponding kinetic energy $T = m(\dot{X}^2 + \dot{Y}^2)/2$ is found to be

$$T = \frac{1}{2}m(\frac{L}{2}\cos\frac{\theta}{2})^2\dot{\theta}^2. \tag{10}$$

Its potential energy $V = mgY$ has the form

$$V = -mg\frac{L}{4}(3 + \cos\theta). \tag{11}$$

On the other hand, a simple pendulum (SP) of the same length and the same mass has a constant period if it performs small oscillations. Its kinetic energy is of the form

$$T_{SP} = \frac{1}{2}m(L\dot{\varphi})^2, \tag{12}$$

and the potential energy is

$$V_{SP} = -mgL(1 - \frac{\varphi^2}{2}). \tag{13}$$

The equality between two expressions for the kinetic energy (10) and (12) yields

$$\dot{\varphi} = \frac{1}{2}\dot{\theta}\cos\frac{\theta}{2}. \tag{14}$$

This is satisfied for

$$\varphi = \sin\frac{\theta}{2}. \tag{15}$$

By using Eq. (15), the expression for the potential energy V (11) becomes

$$V = -mgL(1 - \frac{1}{2}\sin^2\frac{\theta}{2}) = -mgL(1 - \frac{\varphi^2}{2}), \tag{16}$$

i.e. it transforms to the potential energy of the simple pendulum (13). This implies that by using a certain coordinate transformation (given by Eq. (15) herein), the kinetic and potential energy of Huygens' pendulum become equal to that of a simple pendulum.

The question that naturally arises is if this type of transformation can be established between a wider class of 1DOF nonlinear oscillators and a simple harmonic oscillator so that these nonlinear oscillators are isochronous. To answer this question, two classes of dynamical systems are considered subsequently.

3. Transformation approach I

Let us consider 1DOF oscillators whose kinetic energy has a quadratic form in the generalized velocity

$$T(x,\dot{x}) = \frac{1}{2}\tilde{T}(x)\dot{x}^2, \tag{17}$$

where $\tilde{T}(x)$ is a position-dependent coefficient of the kinetic energy, which can stem from the geometry of motion or displacement-dependent mass, and $V(x)$ is the potential energy that is required to be positive definite and to yield the amplitude-independent frequency.
Lagrange's equation corresponding to this conservative system is:

$$\ddot{x} + \frac{\tilde{T}'}{2\tilde{T}}\dot{x}^2 + \frac{V'}{\tilde{T}} = 0, \tag{18}$$

where $\tilde{T}' = d\tilde{T}/dx$. Now, putting the requirement of the equivalence between the kinetic and potential energy of the oscillator under consideration and that of a simple harmonic oscillator (SHO) whose generalized coordinate is labelled by X

$$T_{\mathrm{SHO}} = \frac{\dot{X}^2}{2}, \quad V_{\mathrm{SHO}} = \frac{X^2}{2}, \tag{19a,b}$$

we conclude that the following should be satisfied

$$\dot{X} = \sqrt{\tilde{T}(x)}\,\dot{x}, \quad V(x) = \frac{X^2}{2}. \tag{20a,b}$$

Equation (20a) gives

$$X = \int_0^x \sqrt{\tilde{T}(s)}ds, \tag{21}$$

and Eq. (20b) with Eq. (21) defines the potential energy, so that the equation of motion (18) becomes

$$\ddot{x} + \frac{\tilde{T}'}{2\tilde{T}}\dot{x}^2 + \frac{1}{\sqrt{\tilde{T}}}\left[\int_0^x \sqrt{\tilde{T}(s)}ds\right] = 0. \tag{22}$$

This equation contains a term that is quadratic in $\dot{x}$, and can be related to Eq. (1), where $p(x) = \tilde{T}'/(2\tilde{T})$ and $q(x) = \left[\int_0^x \sqrt{\tilde{T}(s)}ds\right]/\sqrt{\tilde{T}}$. Now, Sabatiniti's results [4] for the isochronicity of its solution can be used to determine the form of $\tilde{T}(x)$. First, it is easy to show that the condition given by Eq. (2) is satisfied. Then, the form of $\tilde{T}(x)$ should be such that $p(x)$ and $q(x)$ are odd and analytic. As $\tilde{T}(x)$ can be mass in classical mechanical systems, given its properties, one concludes that $p(x)$ and $q(x)$ are odd if $\tilde{T}(x)$ is even (note that for the origin to be a center it is required that

$xq(x) > 0$, i.e. $x\left[\int_0^x \sqrt{\tilde{T}(s)}ds\right] / \sqrt{\tilde{T}} > 0$).

Since the solution for motion of the SHO has a general form $A\cos(t+\alpha)$, Eq. (21) also defines how x changes with time

$$X = \int_0^x \sqrt{\tilde{T}(s)}ds = A\cos(t+\alpha), \tag{23}$$

where A and α can be found from the initial conditions $x(0)$ and $\dot{x}(0)$. So, not only does this approach yield mechanical and mathematical models of isochronous oscillators, but it also enables one to find their isochronous motion. In addition, as these systems are conservative, the energy-conservation law $T + V = h$, where h is the corresponding initial energy level, can be used to define isochronous motion in the phase plane as follows

$$\dot{x}^2 = \frac{2h - \left[\int_0^x \sqrt{\tilde{T}(s)}ds\right]^2}{\tilde{T}(x)}. \tag{24}$$

The following example demonstrates some potential benefits of these theoretical findings. A few more examples of 1DOF isochronous oscillators belonging to this class can be found in [9] and [10].

3.1. Example I.1

Let us assume that the coefficient $\tilde{T}$ changes with the displacement as follows

$$\tilde{T}(x) = 1 + \sin^2 x. \tag{25}$$

The integral in Eq. (21) can be expressed in terms of the incomplete elliptic integral of the second kind $E(x|\,m)$, with the elliptic modulus *m* indicated below:

$$X = \int_0^x \sqrt{1+\sin^2 s}\,ds = E(x|-1). \tag{26}$$

The potential energy (20b) is

$$V(x) = \frac{E^2(x|-1)}{2}. \tag{27}$$

The equation of motion has the form

$$\ddot{x} + \frac{\sin 2x}{2(1+\sin^2 x)}\dot{x}^2 + \frac{1}{\sqrt{1+\sin^2 x}}E(x|-1) = 0. \tag{28}$$

Equations (23) and (24) respectively give the solution for motion and phase trajectories:

$$E(|x|-1)=A\cos(t+\alpha),\quad \dot{x}^2=\frac{2h-E^2(|x|-1)}{1+\sin^2 x}. \tag{29a,b}$$

In Figure 2a, time responses obtained numerically from Eq. (28) are plotted as black solid lines for $x(0)=0.5;1;2$ and $\dot{x}(0)=0$, while the corresponding analytical solutions given by Eq. (29a) are shown in red dots. These solutions match and confirm that the motion is isochronous.

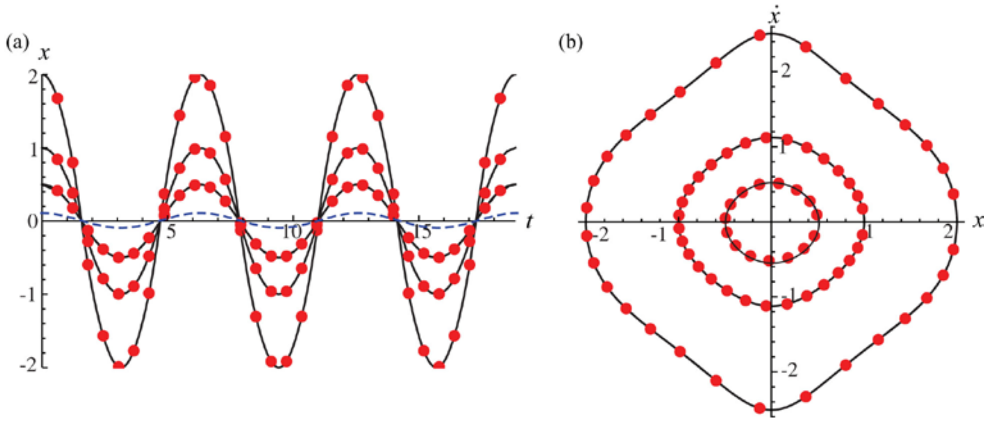

Figure 2. Numerically obtained results from Eq. (28) (black solid line), Eq. (30) (blue dashed line) and the corresponding analytical solution Eq. (29a,b) (red dots) for $x(0)=0.1;\,0.5;\,1;\,2$ and $\dot{x}(0)=0$: a) time responses; b) phase trajectories.

In addition, Figure 2b shows phase trajectories obtained numerically from Eq. (28) and those based on the expression (29b). These solutions also coincide with each other.

Recognising the functional coefficient $p(x)$ and $q(x)$ from Eq. (1) in Eq. (28), one can develop them into series to obtain

$$\ddot{x}+\left(x-\frac{5}{3}x^3+\frac{32}{15}x^5+\ldots\right)\dot{x}^2+x-\frac{1}{3}x^3+\frac{2}{5}x^5+\ldots=0. \tag{30}$$

Given the approximation performed, one can expect that for the case of small oscillations, the corresponding time response will also have a constant period as Eq. (28). This is confirmed in Figure 2a, where the numerical solution of Eq. (30) with $p(x)$ and $q(x)$ truncated to quintic polynomials is shown as a blue dashed line, having the same constant period as other time responses obtained from Eq. (28).

4. Transformation approach II

So far we have considered 1DOF oscillators whose kinetic energy has a pure quadratic form in generalized velocities. However, it is known that the structure of the kinetic energy of holonomic dynamical systems is such that their kinetic energy can be represented as a sum [11]

$$T(t,x,\dot{x}) = T_2(t,x,\dot{x}) + T_1(t,x,\dot{x}) + T_0(t,x), \tag{31}$$

where $T_2(t,x,\dot{x})$ is a quadratic form of the generalized velocity, $T_1(t,x,\dot{x})$ is a linear form of the generalized velocity, while $T_0(t,x)$ does not depend on the generalized velocity. Motivated by this fact, we consider now systems whose kinetic energy has the form

$$T = \frac{1}{2}\left[\dot{x} + x \cdot f(x)\right]^2 \cdot \left[\exp \int_0^t f(x(\tau))d\tau\right]^2. \tag{32}$$

Note that it is easy to verify that Eq. (32) includes all three characteristic terms of the kinetic energy T_2, T_1 and T_0. The equivalence between the kinetic energy (32) and the one of the simple harmonic oscillator (19a) leads to

$$\dot{X} = \left[\dot{x} + x \cdot f(x)\right] \cdot \left[\exp \int_0^t f(x(\tau))d\tau\right]. \tag{33}$$

Equation (33) is satisfied for

$$X = x \cdot \left[\exp \int_0^t f(x(\tau))d\tau\right]. \tag{34}$$

Based on the analogy with the potential energy of the simple harmonic oscillator (19b), the potential energy of the oscillators under consideration is

$$V(x) = \frac{X^2}{2} = \frac{1}{2}x^2 \cdot \left[\exp \int_0^t f(x(\tau))d\tau\right]^2. \tag{35}$$

For the kinetic energy (32) and the potential energy (35), Lagrange's equation of motion is

$$\ddot{x} + (2f + xf')\dot{x} + \left(1 + f^2\right)x = 0. \tag{36}$$

This equation belongs to a class of Liénard-type equations (4), where $u(x) = 2f + x f'$ and $v(x) = 1 + f^2$. The conditions for this equation to have an isochronous solution are listed in the paragraph after Eq. (4). They imply that *f* should be such that $u(x) = 2f + x f'$ and $v(x) = (1 + f^2)x$ are analytic, $v(x) = (1 + f^2)x$ is odd, $u(0) = v(0) = 0$, $v'(0) > 0$, as well as that Eq. (5) is satisfied. After determining *f* that satisfies all these conditions, one can go back to Eqs. (33) and (34), as their combination, together with $X = A\cos(t+\alpha)$, leads to

$$\frac{\dot{x}}{x} + f(x) = -\tan(t+\alpha), \tag{37}$$

If solvable analytically, this equation gives the solution for motion *x*(*t*).

It should be noted that that these systems are characterized by the energy-like conservation law which stems from $\dot{X}^2/2 + X^2/2 = const.$, and has the form:

$$\frac{1}{2}\left([\dot{x} + x \cdot f(x)]^2 + x^2\right) \cdot \left[\exp \int_0^t f(x(\tau))d\tau\right]^2 = const. \tag{38}$$

4.1. Example II.1

It is easy to show that all the conditions for Eq. (36) to exhibit isochronous oscillations around the origin are satisfied if one takes the function $f(x)$ as an odd polynomial function of the form

$$f(x) = \frac{\mu}{2n+3} x^{2n+1} + \frac{\eta}{2n+5} x^{2n+3}, \tag{39}$$

where n is a non-negative integer, and μ, η are certain constants. The corresponding equation of motion is

$$\ddot{x} + \left(\mu x^{2n+1} + \eta x^{2n+3}\right)\dot{x} + x + \frac{\mu^2}{(2n+3)^2} x^{4n+3} + \frac{2\mu\eta}{(2n+3)(2n+5)} x^{4n+5} + \frac{\eta^2}{(2n+5)^2} x^{4n+7} = 0. \tag{40}$$

This equation was solved numerically for different initial conditions and these solutions are plotted in Figure 3. The time histories presented in this figure confirm that the period of vibration is independent of the amplitude.

For $\eta = 0$ and $\mu = 2n+3$, Eq. (40) simplifies to Sabatini's example given by Eq. (6), whose exact solution and some related investigations can be found in [6].

It is also interesting to point out that for

$$f(x) = \frac{\mu}{2n+2} x^{2n}, \tag{41}$$

the corresponding equation of motion takes the form

$$\ddot{x} + \mu x^{2n} \dot{x} + x + \frac{\mu^2}{(2n+1)^2} x^{4n+1} = 0, \tag{42}$$

and the condition (5) is satisfied. However, although the restoring force is odd, the coefficient in front of $\dot{x}$ is not odd, and this condition required by Sabatini's isochronicity conditions is not satisfied.

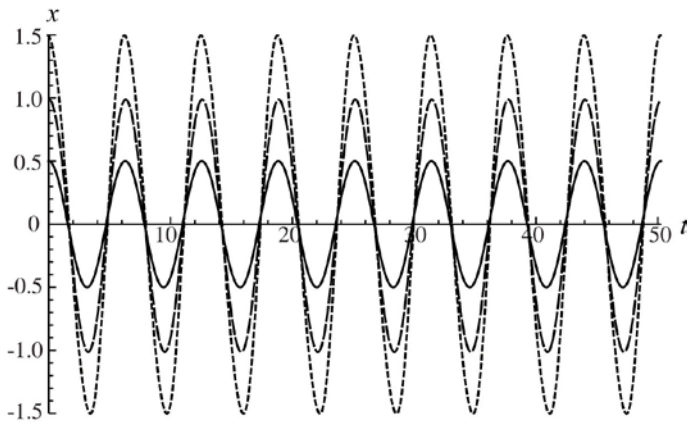

Figure 3. Numerically obtained time responses from Eq. (40) for n=1, μ=η=0.1, $x(0) = 0.5; 1; 1.5$ and $\dot{x}(0) = 0$.

Using Eq. (37), one obtains

$$\frac{\dot{x}}{x} + \frac{\mu}{2n+2} x^{2n} = -\tan(t+\alpha), \tag{43}$$

and its solution for positive values of n is

$$x^{2n} = \frac{n+1}{\mu n} \frac{\cos^{2n}(t+\alpha)}{\int \cos^{2n}(t+\alpha) dt + C_1}, \tag{44}$$

where C_1 is an arbitrary constant.

For n=0, Eq. (42) simplifies to the equation of motion of a linear damped oscillator

$$\ddot{x} + \mu \dot{x} + \left(1 + \mu^2\right) x = 0. \tag{45}$$

The conservation law (38) turns now into the known conservation law for this oscillator [11]:

$$\frac{1}{2}\left(\left[\dot{x}+x\cdot\frac{\mu}{2}\right]^2+x^2\right)\cdot e^{\mu t}=const., \tag{46}$$

while Eq. (45), gives $x = A\cos(t+\alpha)\cdot e^{-\mu t}$. It is well-known that a linear under-damped oscillator has a constant period, which depends on the natural frequency and the damping coefficient, and is usually called a quasi-period. However, its amplitude is not constant, but it decays in time. So, this oscillator has the property of damped isochronicity. Similarly, the oscillator modelled by Eq. (42) can have the same characteristic of damped isochronicity – a constant period and a decaying amplitude. Time histories obtained numerically from Eq. (42) and based on the analytical result (44) for *n*=1 are shown in Figure 4 and confirm this finding.

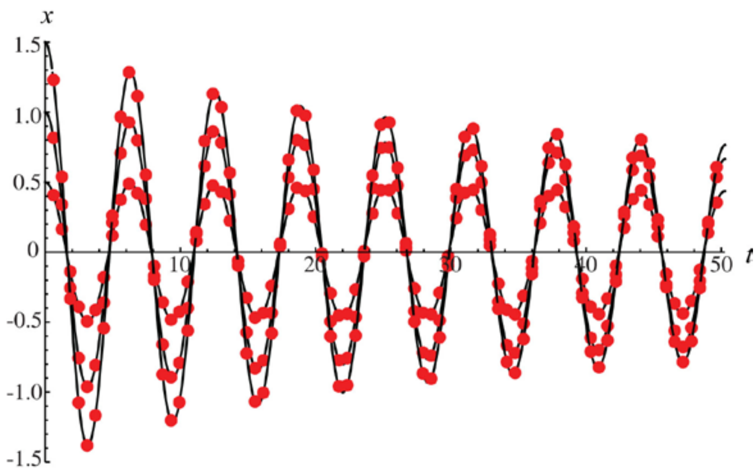

Figure 4. Numerically obtained time responses from Eq. (42) for *n*=1, μ=0.1, $x(0)=0.5;1;1.5$ and $\dot{x}(0)=0$ (black solid line), and the analytical solution plotted based on Eq. (44) (red dots).

5. Conclusions

This study has been concerned with nonlinear oscillators that have isochronous orbits around the origin. Their mechanical and mathematical models have been proposed based on the transformation approach in which the equivalence between their kinetic and potential energy and that of a simple harmonic oscillator is established. Two types of dynamical systems have been considered. The first type corresponds to conservative dynamical systems whose equations of motion contain a term quadratic in the generalized velocity. General expressions for the corresponding isochronous solutions and phase trajectories are derived. The second type is characterized by the equations of motion with a term linear in the generalized velocity. The corresponding energy-conservation law has been derived and the property of isochronicity and damped isochronicity has been demonstrated.

Acknowledgement

Ivana Kovacic acknowledges support received from the Provincial Secretariat for Science and Technological Development, Autonomous Province of Vojvodina, Republic of Serbia (Project No. 114-451-2094).

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