Dynamics of a mass-spring-pendulum system with vastly different frequencies

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Abstract. *We investigate the dynamics of a simple pendulum coupled to a horizontal mass-spring system. The spring is assumed to have a very large stiffness value such that the natural frequency of the mass-spring oscillator, when uncoupled from the pendulum, is an order of magnitude larger than that of the oscillations of the pendulum. The leading order dynamics of the autonomous coupled system is studied using the method of Direct Partition of Motion(DPM), in conjunction with a rescaling of fast time in a manner that is inspired by the WKB method. We particularly study the motions in which the amplitude of the motion of the harmonic oscillator is an order of magnitude smaller than that of the pendulum. In this regime, a pitchfork bifurcation of periodic orbits is found to occur for energy values larger that a critical value. The bifurcation gives rise to non-local periodic and quasi-periodic orbits in which the pendulum oscillates about an angle between zero and* $\pi/2$ *from the down right position. The bifurcating periodic orbits are nonlinear normal modes of the coupled system and correspond to fixed points of a Poincare map. An approximate expression for the value of the new fixed points of the map is obtained. These formal analytic results are confirmed by comparison with numerical integration.*

Keywords: *Coupled oscillators, DPM, Method of direct partition of motion, WKB method, Bifurcations*

1. INTRODUCTION

The method of direct partition of motion (DPM), formalized by Blekhman (Blekhman, 2000), serves to facilitate the study of the non-trivial effects of fast excitation that has been elaborately studied in recent years (Blekhman, 2000), (Jensen, 1999), (Thomsen 2003, 2005). In most problems addressed in the literature, the fast excitation is due to an external source, that is, the system considered is non-autonomous. However, similar non-trivial effects could occur even if the fast excitation is internal to the system, instead of coming from an external source. An example of such a case would be a nonlinear oscillator coupled to a much faster oscillator (Tuwankotta and Verhulst, 2003), (Nayfeh and Chin, 1995), (Sheheitli and Rand, 2011). In these latter autonomous systems with widely fast frequencies, the leading order dynamics of the fast oscillator is unaffected by the slow oscillator. This is not the case for the system we study in this paper, as the amplitude and frequency of the fast oscillation, to leading order, are found to be a function of the amplitude of the slow oscillation. This is established by observing that the equation of the fast degree of freedom can be treated as a fast oscillator with a slowly varying frequency, for which the WKB method is particularly suited, and thus using a transformation of fast time analogous to that proposed by the WKB method (Wilcox, 1995).

2. THE MASS-SPRING-PENDULUM SYSTEM

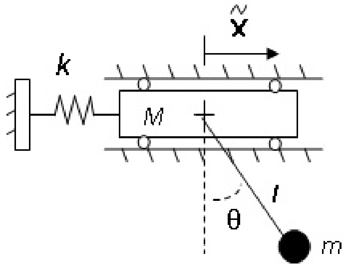


Figure 1: Schematic for the mass-spring-pendulum system

We consider a simple pendulum whose point of suspension is connected to a mass on a spring that is restricted to move horizontally, as shown in Fig.1. Ignoring dissipation, the system is governed by the following equations of motion:

$$ml^2\theta'' + ml\tilde{x}''\cos\theta + mgl\sin\theta = 0$$

$$(M+m)\,\tilde{x}'' + ml\theta''\cos\theta - ml\theta'^2\sin\theta + k\tilde{x} = 0$$

where primes denotes differentiation with respect to time τ. We introduce the following change of variables:

$$x = \tilde{x}\big/l \quad , \quad t = \tau\sqrt{g/l}$$

The nondimensionalized equations, which we refer to as the full system, become:

$$\ddot{\theta} + \sin\theta = -\ddot{x}\cos\theta$$

$$\ddot{x} + \tilde{\Omega}^2 x = -\mu\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) \quad where \quad \mu = \frac{m}{M+m} \quad , \quad \tilde{\Omega}^2 = \frac{kl}{g\,(M+m)} \tag{1}$$

where dots represent differentiation with respect to time t.

2.1 Assumptions

We are interested in the case where the linear oscillator has a natural frequency that is an order of magnitude larger than the linearized frequency of the pendulum, and its motion has an amplitude that is an order of magnitude smaller than that of the pendulum. This is implemented through the following rescaling: $x = \varepsilon\chi$, $\tilde{\Omega}^2 = \Omega^2\big/\varepsilon^2$. Here, Ω and χ are $O\,(1)$ quantities while $\varepsilon << 1$. Without loss of generality, we take $\Omega = 1$. The rescaled equations become:

$$\ddot{\theta} + \sin\theta = -\varepsilon\ddot{\chi}\cos\theta$$

$$\ddot{\chi} + \frac{1}{\varepsilon^2}\chi = -\frac{\mu}{\varepsilon}\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) \tag{2}$$

This system has a conserved quantity that can be expressed as:

$$h = \frac{1}{2}\varepsilon^2\dot{\chi}^2 + \frac{1}{2}\mu\dot{\theta}^2 + \mu\varepsilon\dot{\chi}\dot{\theta}\cos\theta + \frac{1}{2}\chi^2 + \mu\,(1-\cos\theta)$$

2.2 The bifurcation of periodic orbits

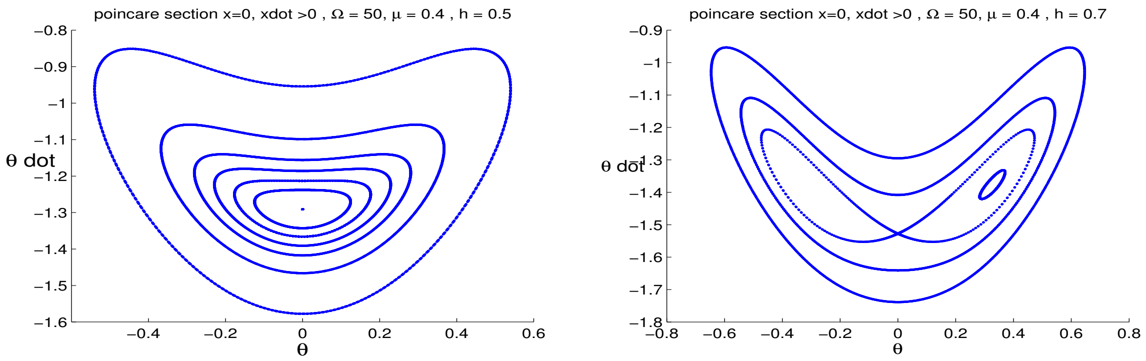
We numerically integrate the full system in Eqs.(1), for typical parameter values and initial conditions (IC's), in order to illustrate the bifurcation. As an example, we take $\mu = 0.4$ and $\tilde{\Omega} = 50$ $(\varepsilon = 0.02)$. Since the system is conservative, we will look at how the dynamics change as the energy is increased. Fig.2a shows the Poincare map for the energy level $h = 0.5$. The shown fixed point (center) of the map corresponds to a periodic orbit in which $\theta \approx -x$. This periodic orbit is a nonlinear normal mode of the coupled system that appears as a nearly straight line through the origin if viewed in the configuration plane θ vs. x. Fig.2b shows the Poincare map for $h = 0.7$. We can see from the Poincare map that the fixed point corresponding to the nonlinear normal mode with $\theta \approx -x$ has lost stability and is now a saddle point of the map. Consequently, we can predict that two new fixed points (centers) were born in the process, and closed orbits of the map about the new centers would correspond to oscillations of the pendulum about a non-zero angle. The aim of this paper is to shed light on these latter non-trivial solutions, in which the pendulum oscillates about a non-zero angle, and describe their dependence on initial conditions and the parameter μ.

3. THE APPROXIMATE SOLUTION

In Eqs.(2), each of the equations contains the second derivative of both χ and θ. We can rewrite the system of equations so that each second derivative appears in only one of the equations, as follows:

$$\ddot{\theta} + \frac{1}{1-\mu\cos^2\theta}\left(\sin\theta + \mu\dot{\theta}^2\cos\theta\sin\theta - \frac{1}{\varepsilon}\chi\cos\theta\right) = 0$$

$$\ddot{\chi} + \frac{1}{1-\mu\cos^2\theta}\left(\frac{1}{\varepsilon^2}\chi - \frac{\mu}{\varepsilon}\left(\dot{\theta}^2\sin\theta + \cos\theta\sin\theta\right)\right) = 0$$



(a) (b)

Figure 2: Poincare map $x = 0, \dot{x} > 0$ for: (a) $h = 0.5$, (b) $h = 0.7$

In this latter form, the θ equation appears as that of a nonlinear oscillator parametrically forced by χ, which we expect to be a fast oscillation. Hence this suggests the partitioning of the θ motion into a slow component overlaid by a fast component, as in the DPM ansatz (Blekhman, 2000). Also, we can see that the χ equation appears as that of a fast oscillator with a frequency whose magnitude is modulated by θ which we expect to be a slow oscillation, that is, it appears as an equation of a fast oscillator with a slowly changing frequency, similar to that which the WKB method (Wilcox, 1995) is well suited for. This suggests rescaling fast time in the following manner:

$$\frac{dT}{dt} = \frac{\omega(t)}{\varepsilon} \qquad \text{or} \qquad T = \int_0^t \frac{\omega(t')}{\varepsilon} dt'$$

and the assumed solution becomes:

$$\begin{cases} \chi = \chi(t, T) \\ \theta(t, T) = \theta_0(t) + \varepsilon\theta_1(t, T) \end{cases}$$

Here, $\omega(t)$ is to be chosen such that the fast oscillation is a perturbation of a harmonic oscillation on the new timescale T. That is, we will choose $\omega(t)$ so that the χ equation has the form:

$$\frac{\partial^2\chi}{\partial T^2} + \chi + O(\varepsilon) = 0 \qquad \text{where} \qquad \omega(t) = \frac{1}{\sqrt{1 - \mu\cos^2\theta_0}}$$

then, an approximate expression for χ can be found using regular perturbations. After applying the standard DPM procedure (Blekhman, 2000), we find that, to leading order, θ_0 is governed by the following equation (see (Sheheitli and Rand, to appear) for the details):

$$\frac{d^2\theta_0}{dt^2} + \sin\theta_0 - \frac{1}{2}C^2 \frac{\cos\theta_0 \sin\theta_0}{(1 - \mu\cos^2\theta_0)\sqrt{1 - \mu\cos^2\theta_0}} = 0 \tag{3}$$

θ_1 is found to be $\theta_1 = -\chi\cos\theta_0$ and, to leading order, χ is given by $\chi \approx C\sqrt{\omega(t)}\cos T$ (see (Shehetli and Rand, to appear) for the details), where C is an arbitrary constant that depends on initial conditions. Then the motion of the pendulum, in the rescaled system described by Eqs.(2), can be expressed as:

$$\theta \approx \theta_0 - \varepsilon\chi\cos\theta_0$$

Recall that Eqs.(2) are a rescaled version of the original system of interest given by Eqs.(1), where χ is related to the motion of the mass-spring oscillator as follows: $x = \varepsilon\chi$. Hence, the solution to Eqs.(1), for the assumed regime of motion, can be expressed in terms of the variables of Eqs.(1) as follows:

$$\theta \approx \theta_0 - x\cos\theta_0 \ , \ \ x \approx \varepsilon C\sqrt{\omega(t)}\cos T \tag{4}$$

4. THE SLOW DYNAMICS

At the end of the DPM procedure that is described in (Sheheitli and Rand, to appear), the solution to the two degree of freedom mass-spring-pendulum system is expressed in Eqs.(4) in terms of θ_0, the leading order slow motion of the pendulum, which is governed by Eq.(3). The arbitrary constant C that appears in the equation can be expressed in terms of the initial conditions. For initial zero velocities, the initial conditions take the form:

$$\left\{\begin{array}{l}\dot{\theta}(0)=0\\ \theta(0)=A\end{array}\right. \quad , \quad \left\{\begin{array}{l}\dot{x}(0)=0\\ x(0)=\varepsilon B\end{array}\right. \Rightarrow \left\{\begin{array}{l}\dot{\theta}_0(0)=0 \quad , \quad \theta_0(0)\approx A\\ \\ x(0)=\varepsilon B\approx \varepsilon C\sqrt{\omega(0)}\end{array}\right. \Rightarrow C=B\left(1-\mu\cos^2 A\right)^{\frac{1}{4}}$$

Now, we rewrite the equation governing θ_0 as a system of two first order equations:

$$\dot{\theta}_0=\phi \ , \ \dot{\phi}=-\sin\theta_0+\frac{1}{2}C^2\frac{\sin\theta_0\cos\theta_0}{\left(1-\mu\cos^2\theta_0\right)\sqrt{1-\mu\cos^2\theta_0}} \tag{5}$$

For small enough values of C, the above system has a neutrally stable equilibrium point (center) at the origin ($\phi=0,\theta_0=0$) and two saddle points at ($\phi=0,\theta_0=\pi,-\pi$), so that the phase portrait resembles that of the simple pendulum. As C increases in value, a pitchfork bifurcation takes place, in which the origin becomes a saddle point and two new centers are born. The critical value of C is related to the parameter μ as follows (see (Sheheitli and Rand, to appear) for the details):

$$C_{cr}^2=2\left(1-\mu\right)^{\frac{3}{2}} \tag{6}$$

In (Sheheitli and Rand, to appear), it is explained how this condition on C translates into the following condition on the energy value h:

$$h_{cr}=1-\mu$$

4.1 The predicted nonlinear normal modes

Note that each value of C leads to a phase portrait filled with closed orbits, however, out of those orbits, the only one which corresponds to a solution of the full system (1) is that associated with the specific IC's that led to that value of C. An interesting case occurs when the choice of IC's results in a phase portrait that has a non-trivial equilibrium point which coincides in value with the initial θ_0 amplitude, A. That is, we start with IC's of the form:

$$\left\{\begin{array}{l}\dot{\theta}(0)=0\\ \theta(0)=\theta_0(0)=A\end{array}\right. \quad , \quad \left\{\begin{array}{l}\dot{x}(0)=0\\ x(0)=\varepsilon B\end{array}\right.$$

and the corresponding value of C results in non-trivial equilibrium points (centers) for the θ_0 equation at:

$$\theta_0=\pm E \ , \ \phi=0$$

Then, if $E=A$, θ_0 will remain equal to E for all time. It would mean that we are starting at a neutrally stable equilibrium point of the θ_0 equation, so the solution will remain at that point for all time.

In (Sheheitli and Rand, to appear), it is shown that these special values of initial θ amplitude can be expressed in terms of h and μ as:

$$\theta(0)=\pm A^*=\pm\cos^{-1}\left(\frac{\mu-h\pm\sqrt{(h-\mu)^2+8\mu}}{4\mu}\right) \tag{7}$$

The corresponding value of x is expressed as:

$$x(0)=\varepsilon B^*=\varepsilon\sqrt{2\left(h-\mu\left(1-\cos A\right)\right)} \tag{8}$$

Hence, we predict that these special initial amplitudes, with zero initial velocities, will lead to a solution in which:

$$\theta\approx A^*-x\cos A^* \tag{9}$$

Such a solution would be a nonlinear normal mode of the coupled mass-spring-pendulum system and corresponds to a non-trivial fixed points of the Poincare map.

4.2 Relation of θ_0 to the Poincare map

For a given energy level, the phase portrait of the θ_0 equation is filled with closed orbits and the picture is topologically similar to that of the Poincare map. That is, for a given initial condition, the resulting orbit in the θ_0 phase plane corresponds to a closed orbit in the Poincare map, however, the orbits are not identical. This is due to the fact that, while $\theta \approx \theta_0$, $\dot{\theta}$ differs from $\dot{\theta}_0$ by an $O(1)$ quantity; as shown in (Sheheitli and Rand, to appear), for the points of the Poincare map, $\dot{\theta}$ can be expressed in terms of $\dot{\theta}_0$ as follows:

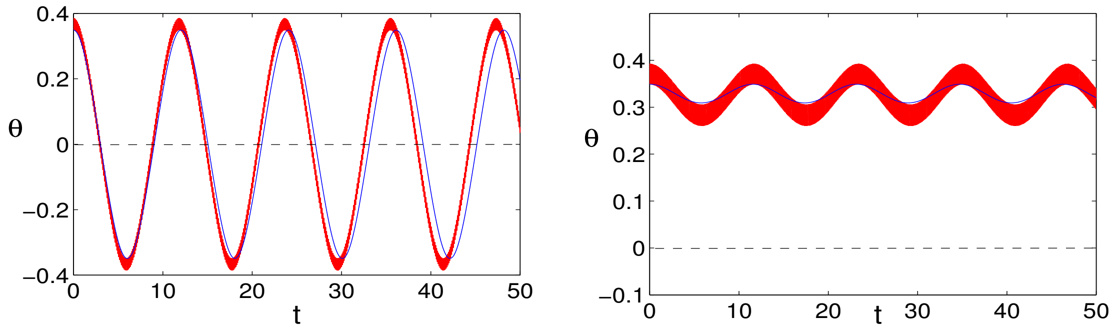
$$\dot{\theta}_{Pm} \approx \dot{\theta}_0 - C\left(1 - \mu \cos^2 \theta_0\right)^{-\frac{3}{4}} \cos \theta_0 \tag{10}$$

So for given IC's, we can obtain the corresponding orbit in the Poincare map by first numerically integrating the θ_0 equation to obtain θ_0 and $\dot{\theta}_0$ and then generating the orbit in the Poincare map by plotting the corresponding values of $\dot{\theta}_{Pm}$ vs. θ_0. This means that we can generate an approximate picture of the Poincare map of the full system (1) by numerically integrating the slow dynamics equation governing θ_0 instead of integrating the full system (1) which contains the fast dynamics and thus requires a much smaller step size of integration.

Also, by comparing this procedure with the results of numerical integration, we can obtain a check on the accuracy of the various approximations made in this work.

5. COMPARISON TO NUMERICS

We compare the solution resulting from the numerical integration of the original equations with that from the integration of the θ_0 equation. We have set $\mu = 0.4$ and $\tilde{\Omega} = 50$ $(\varepsilon = 0.02)$. In the plots of Fig.3, the thick line correspond



(a) (b)

Figure 3: Comparison plots of θ vs. time for IC's with (a) A=π/9 , B=0.9756 (h=0.5) (b) A=π/9 , B=1.1626 (h=0.7)

to θ vs. time for the solution of the numerical integration of the full system (1), and the apparent thickness is due to the fast component present in the oscillation of the pendulum; the thin line corresponds to the approximate solution, that is from the numerical integration of the θ_0 equation, and captures only the leading order slow component of the pendulum oscillation. Fig.4 displays the Poincare map orbits for $h = 1$. Near each of the orbits, a small arrow points to the orbit which is predicted from the θ_0 equation for corresponding initial conditions. We can see that that the approximate solution compares well with that from numerical integration of the full system (1).

6. CONCLUSION

We have used the method of direct partition of motion to study the dynamics of a mass-spring-pendulum system, in which the harmonic oscillator is restricted to move horizontally. We have considered the case where the stiffness of the spring is very large, so that the frequency of the oscillation of the uncoupled harmonic oscillator is an order of magnitude larger than that of the uncoupled pendulum. We have also limited our attention to the regime of motion where the amplitude of motion of the harmonic oscillator is an order of magnitude smaller than that of the pendulum. Under these assumptions, an approximate expression for the solution of the two degree of freedom system is found in terms of θ_0, the leading order slow oscillation of the pendulum. An equation governing θ_0 is presented and found to undergo a pitchfork bifurcation for a critical value of C which is a parameter related to the initial amplitudes of θ and x. It is shown

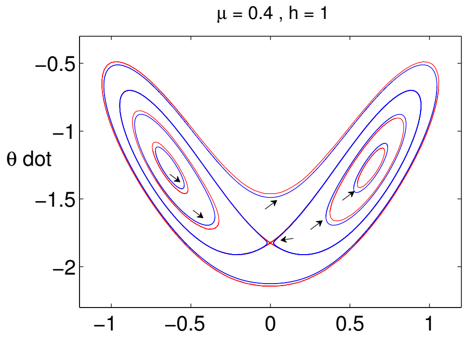


Figure 4: Comparison of the predicted Poincare map orbits (arrows) with those from the integration of the full system (1)

that the pitchfork bifurcation in the slow dynamics equation corresponds to a pitchfork bifurcation of periodic orbits of the full system (1) that occurs as the energy is increased past a critical value which is expressed in terms of the parameter μ. This bifurcation can be seen to occur in the Poincare map of the full system (1), where the fixed point corresponding to the nonlinear normal mode $\theta \approx -x$ loses stability and two new centers are born in the map. The new centers correspond to new periodic motions, which are nonlinear normal modes with $\theta \approx A^* - x \cos A^*$, where the expression for A^* is found in terms of μ and h. For these modes, the motion of the pendulum is predicted to be a small fast oscillation about the non-zero value $\theta = A^*$. Along with these special motions, quasi-periodic motions exist in which the pendulum undergoes slow oscillation about a non-zero angle, with overlaid fast oscillation. These latter orbits correspond to closed orbits about the new centers in the Poincare map. A relation between $\dot{\theta}$ and $\dot{\theta}_0$ is given for points of the Poincare map, such that the orbits of the map can be generated approximately by numerically integrating the slow dynamics equation. Finally, the approximate solution, as well as the predications made based on it, are checked against numerical integration of the full system (1) and found to agree well.

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