



COMPUTER ALGEBRA, ELLIPTIC FUNCTIONS AND CHAOS

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ABSTRACT

We generalize the method of averaging to perturb off of nonlinear oscillators of the form:

$$\ddot{x} + \alpha x + \beta x^3 = 0$$

where α and β are given constants. This permits us to study the following class of systems:

$$\ddot{x} + \alpha(\epsilon t) x + \beta(\epsilon t) x^3 = \epsilon g(x, \dot{x}, \epsilon t), \quad \epsilon \ll 1$$

where ϵt represents slow time and where $\alpha(\epsilon t)$ and $\beta(\epsilon t)$ are slowly varying parameters.

The computations required for this work involve the algebraic manipulation of the Jacobian elliptic functions cn, sn and dn . We use computer algebra (MACSYMA) to perform these computations.

In particular, this approach is applied to the problem:

$$\ddot{x} - x \cos \epsilon t + x^3 = 0$$

which is shown to exhibit chaotic behavior. The slow forcing permits us to envision an "instantaneous phase portrait" in the $x-\dot{x}$ phase plane which exhibits a center at the origin when $\cos \epsilon t \leq 0$ and a saddle and associated double homoclinic loop separatrix when $\cos \epsilon t > 0$. The chaos in this problem is related to the question of on which side (left (=L) or right (=R)) of the periodically reappearing double homoclinic loop separatrix a motion finds itself. We show that the sequence of L's and R's exhibits sensitive dependence on initial conditions. The investigation of chaos is accomplished without the use of the well-known Melnikov's method, which cannot be applied here since the separatrix periodically disappears. It is shown that this model is applicable to the dynamics of the familiar rotating-plane pendulum.

INTRODUCTION

This work involves using a perturbation method, averaging, to study systems which are close to the strongly nonlinear oscillator:

$$(1) \quad \ddot{x} + \alpha x + \beta x^3 = 0$$

Eq.(1) possesses a general solution involving the Jacobian elliptic functions cn, sn and dn .

Although the method of averaging has been treated in numerous references, most treatments deal almost exclusively with perturbations off of the simple harmonic oscillator. A few authors (see literature review in next section) have considered perturbations off of eq.(1), generally obtaining expressions for the approximate equations of motion in terms of integrals involving elliptic functions. However, little use has been made of these treatments because of the complicated algebraic manipulations required for the evaluation of the integrals. By using the computer algebra system MACSYMA, we have been able to efficiently evaluate the associated integrals.

The work reported in this paper has been described more fully in Coppola (1989), and the reader is referred there for details. In particular, the listing of the MACSYMA program is given there.

LITERATURE REVIEW

The following works involve perturbations using elliptic functions: Kuzmak (1959) looks for periodic solutions in eq.(1) using a multiple scale method, where α and β are slowly varying parameters. Davis (1962) investigates second order ordinary differential equations using elliptic functions. Cap (1973) applies the method of averaging to perturbations of the mathematical pendulum. Pocobelli (1981) studies the slowly varying pendulum. Greenspan and Holmes (1983) and Guckenheimer and Holmes (1983) apply the Melnikov method to perturbations of eq.(1) where $\alpha < 0$. Garcia-Margallo and Bejarano (1989) find limit cycles in a generalized Van der Pol oscillator using generalized harmonic balance. Nayfeh (1973), Kevorkian

and Cole (1981) and Sanders and Verhulst (1985) also treat such problems.

The following works involve perturbations of eq.(1) with $\alpha = 0$ (a purely nonlinear oscillator): Chirikov (1979) studies resonance overlap under multiple harmonic excitations. Yuste and Bejarano (1986) use first order averaging as a means to find transitory behavior as the motion is attracted to a limit cycle. Garcia-Margallo and Bejarano (1987) employ generalized harmonic balance in order to approximate limit cycles.

ELLIPTIC FUNCTIONS

Since some readers may be unfamiliar with Jacobian elliptic functions, we offer the following brief introduction (see Byrd and Friedman (1954)). The functions cn and sn are the elliptic counterparts of the trigonometric functions \cos and \sin . Like \cos , cn is even, while like \sin , sn is odd. The identity

$$(2) \quad sn^2 + cn^2 = 1$$

reminds us of the corresponding relation between \sin and \cos . The functions cn and sn are actually a family of functions which are parameterized by a "modulus" k :

$$(3) \quad cn = cn(u,k), \quad sn = sn(u,k)$$

In fact, cn and sn reduce to \cos and \sin for $k = 0$. The derivatives of cn and sn , however, introduce a new function, dn , which has no trigonometric counterpart:

$$(4) \quad \frac{d}{du} cn = -sn \, dn, \quad \frac{d}{du} sn = cn \, dn$$

In the trigonometric limit $k = 0$, dn reduces to unity. The function dn satisfies the following equations:

$$(5) \quad k^2 sn^2 + dn^2 = 1, \quad \frac{d}{du} dn = -k^2 sn \, cn$$

The functions cn and sn are periodic with period $4 K(k)$, where $K(k)$ is a tabulated function called the complete elliptic integral of the first kind. For k going from 0 to 1, $K(k)$ goes from $\pi/2$ to infinity. The function dn has period $2 K(k)$.

THE UNPERTURBED PROBLEM

Although the general solution to eq.(1) may be found by utilizing the conservation of energy, evaluating the resulting elliptic integral and inverting, we find it more instructive to proceed as follows: Assume a solution in the form:

$$(6) \quad x = r \, cn(u,k), \quad u = a \, t + b$$

where r, a, b and k are constants. Differentiating (6) and using (4),

$$(7) \quad \dot{x} = -r \, a \, cn' = -r \, a \, sn \, dn$$

Differentiating (7) and using (2)-(5) gives

$$(8) \quad \begin{aligned} \ddot{x} &= -r \, a^2 (sn' \, dn + sn \, dn') \\ &= -r \, a^2 (cn \, dn^2 - k^2 \, cn \, sn^2) \\ &= -r \, a^2 ([1-2k^2] \, cn + 2k^2 \, cn^3) \end{aligned}$$

Substituting (8) into (1) gives:

$$(9) \quad [ra^2(2k^2-1)+ar] \, cn + [-2k^2ra^2+\beta r^3] \, cn^3 = 0$$

Equating the coefficients of cn and cn^3 in (9) to zero, we find

$$(10) \quad a^2 = \alpha + \beta \, r^2, \quad k^2 = \frac{\beta \, r^2}{2 \, a^2}$$

The general solution to (1) is therefore given by (6), in which r and b are arbitrary constants, and where a and k are given by (10).

An alternate form of the general solution is:

$$x = r \, dn(u,k), \quad u = a \, t + b$$

However, the identity $cn(u,k) = dn(ku,1/k)$ obviates the need to consider this form of the solution. The reader is referred to Coppola (1989) for an extensive discussion of modulus transformations.

VARIATION OF PARAMETERS

We shall use the solution (6) as a starting point for a perturbation solution of the equation

$$(11) \quad \ddot{x} + \alpha(\epsilon t) \, x + \beta(\epsilon t) \, x^3 = \epsilon \, g(x, \dot{x}, \epsilon t)$$

where $\epsilon \ll 1$. In (11), ϵt represents slow time and $\alpha(\epsilon t)$ and $\beta(\epsilon t)$ are slowly varying parameters. The first step will be to use variation of parameters to express the influence of the slowly varying and order ϵ terms in (11) on the solution (6) of the unperturbed equation. In contrast to the method of averaging, the computations presented in this section are exact. However, the results are intractable and unenlightening. The method of averaging (introduced in the next section) replaces the results obtained in this section by more useful equations, which are, however, approximate (valid in the small ϵ limit.)

In eq.(6), r and b are arbitrary constants of integration. As usual in the method of variation of parameters, we look for a solution to eq.(1) of the form of eq.(6), where the two arbitrary constants r and b are allowed to vary in time. This results in first order differential equations on $r(t)$ and $b(t)$. Unfortunately the resulting expression for $db(t)/dt$ is not periodic, and is thus unsuitable for averaging. This may be remedied by replacing b by φ , where

$$(12) \quad u = 4 \, K(k) \, \varphi$$

The proportionality factor $4 \, K(k)$ in eq.(12) takes into account the dependence of frequency on amplitude in eq.(1), thus eliminating "phase shear". The resulting equations on r and φ have been derived by using computer algebra. Writing $\tau = \epsilon t$, they are:

$$(13.1) \quad \begin{aligned} \frac{dr}{d\tau} &= -\epsilon \, g \frac{sn \, dn}{\sqrt{\alpha + \beta \, r^2}} - \epsilon \frac{d\alpha}{d\tau} \frac{r \, (1 - cn^2)}{2 \, (\alpha + \beta \, r^2)} \\ &\quad - \epsilon \frac{d\beta}{d\tau} \frac{r^3 \, (1 - cn^4)}{4 \, (\alpha + \beta \, r^2)} \end{aligned}$$

$$(13.2) \quad \frac{d\varphi}{dt} = \frac{\sqrt{\alpha + \beta r^2}}{4K} + \epsilon g \frac{2\alpha(\alpha + \beta r^2)(Z \operatorname{sn} \operatorname{dn} + \operatorname{cn}) + \alpha\beta r^2 \operatorname{cn}^3 + \beta^2 r^4 \operatorname{cn}}{4K r (\alpha + \beta r^2)^{3/2} (2\alpha + \beta r^2)} + \epsilon \frac{d\alpha}{dr} \frac{\alpha Z \operatorname{cn}^2 - (2\alpha + \beta r^2) Z - \alpha \operatorname{cn} \operatorname{sn} \operatorname{dn}}{4K (\alpha + \beta r^2) (2\alpha + \beta r^2)} + \epsilon \frac{d\beta}{dr} \frac{[\alpha\beta r^2(Z \operatorname{cn}^4 + Z - \operatorname{cn} \operatorname{sn} \operatorname{dn} [2 + \operatorname{cn}^2]) + 2\alpha^2 Z - \beta^2 r^4 \operatorname{cn} \operatorname{sn} \operatorname{dn}]}{8K \beta (\alpha + \beta r^2) (2\alpha + \beta r^2)}$$

where Z is the Jacobi zeta function:

$$(14) \quad Z(u) = E(u) - \frac{E(k)}{K(k)} u$$

in which $E(u)$ and $E(k)$ are respectively the incomplete and complete elliptic integrals of the second kind. $Z(u)$ is periodic with period $2K(k)$ and has zero mean. Eqs.(13) are periodic in φ and are thus in the proper form for averaging.

THE METHOD OF AVERAGING

In order to explain the method of averaging (Sanders and Verhulst (1985)), we write eqs.(13) in the abbreviated form:

$$(15.1) \quad \dot{r} = \epsilon F_1(r, \varphi)$$

$$(15.2) \quad \dot{\varphi} = \Omega(r) + \epsilon F_2(r, \varphi)$$

The method is based on positing a near-identity transformation from (r, φ) to $(\bar{r}, \bar{\varphi})$:

$$(16.1) \quad r = \bar{r} + \epsilon w_1(\bar{r}, \varphi) + \epsilon^2 v_1(\bar{r}, \varphi) + O(\epsilon^3)$$

$$(16.2) \quad \varphi = \bar{\varphi} + \epsilon w_2(\bar{r}, \varphi) + O(\epsilon^2)$$

where the generating functions w_1 , v_1 and w_2 are to be chosen so as to simplify the resulting equations. Substituting eqs.(16) into eqs.(15) and collecting terms gives equations of the form:

$$(17.1) \quad \dot{\bar{r}} = \epsilon [F_1(\bar{r}, \varphi) - \Omega(\bar{r}) \frac{\partial w_1}{\partial \varphi}] + \epsilon^2 [H_1(\bar{r}, \varphi) - \Omega(\bar{r}) \frac{\partial v_1}{\partial \varphi}]$$

$$(17.2) \quad \dot{\bar{\varphi}} = \Omega(\bar{r}) + \epsilon [F_2(\bar{r}, \varphi) + \frac{d\Omega}{dr}(\bar{r}) - \Omega(\bar{r}) \frac{\partial w_2}{\partial \varphi}]$$

The generating functions w_1 , v_1 and w_2 are chosen so that eqs.(17) are in averaged form, i.e.,

$$(18.1) \quad \dot{\bar{r}} = \epsilon \bar{F}_1(\bar{r}) + \epsilon^2 \bar{H}_1(\bar{r})$$

$$(18.2) \quad \dot{\bar{\varphi}} = \Omega(\bar{r}) + \epsilon \bar{F}_2(\bar{r})$$

where \bar{F}_1 , \bar{F}_2 , and \bar{H}_1 are the means of F_1 , F_2 , and H_1 taken over one period in the periodic variable φ .

We have written a computer algebra (MACSYMA) program that implements the averaging procedure (15)-(18) for eqs.(13). (See Rand and Armbruster (1987) for an introduction to MACSYMA.) The user first inputs expressions for α , β and g , which may contain symbolic parameters. The computer then generates F_1 , F_2 , and Ω . Using an elliptic function integration subroutine that we developed, the program then finds \bar{F}_1 , \bar{F}_2 , w_1 , and w_2 . The program then proceeds to second order averaging of the r equation. We found it essential to proceed in several steps in order to prevent excessive intermediate expression swell. First, H_1 is computed and its terms are divided up among several pre-identified categories. These categories group together terms whose means are computed in like manners. The mean of H_1 is then computed category by category. After this second order averaging is completed, the program outputs the averaged system and the first order transformation.

The MACSYMA program consists of 410 lines of code. Typical runs on a Symbolic 3670 computer take from one to six hours. For example, we found that for g consisting of three terms (\dot{x} , $x \dot{x}^2$, \dot{x}^3), there were 657 second order terms to be averaged. For more information on the program, see Coppola (1989) which contains many applications.

ROTATING PENDULUM

In order to motivate the example which we will treat, we consider the familiar rotating-plane pendulum, Fig.1. A plane pendulum is forced to rotate about a vertical axis with given angular velocity $\omega(t)$. The Lagrangian is

$$(19) \quad L = \frac{m}{2} [R \dot{\psi}^2 + R^2 \omega(t)^2 \sin^2 \psi] + m g R \cos \psi$$

where m is the mass of the pendulum, R is its length and g is the acceleration of gravity. The corresponding equation of motion is

$$(20) \quad \ddot{\psi} + \left[\frac{g}{R} - \omega(t)^2 \cos \psi \right] \sin \psi = 0$$

As is well known, this system, for fixed ω , possesses 1 or 3 equilibria depending respectively on whether ω^2 is smaller or larger than g/R . We choose $\omega(t)$ to be periodically sometimes larger and sometimes smaller than g/R . In order to simplify eq.(20), we use the following scalings:

$$(21) \quad \omega^2 = \frac{g}{R} + \epsilon \cos(\epsilon^{3/2} t), \quad \psi = \sqrt{2R\epsilon/g} x, \quad t = \eta/\sqrt{\epsilon}$$

Substituting eqs.(21) into eq.(20) and expanding for small ϵ , we obtain

$$(22) \quad \frac{d^2 x}{d\eta^2} - x \cos(\epsilon\eta) + x^3 + \dots = 0$$

where \dots represents terms of order ϵ .

Shaw and Wiggins (1988) have investigated chaos in a similar system by using Melnikov's method. This procedure is inapplicable to the present problem, however, since the separatrix periodically disappears.

PERIODICALLY DISAPPEARING SEPARATRIX

For convenience, we rewrite eq.(22) by replacing η by t and neglecting terms of order ϵ :

$$(23) \quad \ddot{x} - x \cos \tau + x^3 = 0, \quad \tau = \epsilon t, \quad \epsilon \ll 1$$

This is a slowly varying Hamiltonian system for which the "instantaneous phase portrait" (i.e. the phase portrait for a fixed value of the forcing function) changes qualitatively with time. For τ fixed, the phase space is filled with closed orbits given by the Hamiltonian

$$(24) \quad \frac{1}{2} \dot{x}^2 - \frac{1}{2} x^2 \cos \tau + \frac{1}{4} x^4 = h$$

The value of h determines which "instantaneous energy curve" a motion is located on.

The sign of $\cos \tau$ determines the qualitative nature of the instantaneous phase portrait which contains either a center at the origin or a double homoclinic loop with saddle at the origin and centers to the left and right, see Fig.2. For $\cos \tau < 0$, all orbits are closed and encircle the center at the origin. For $\cos \tau > 0$, there are three regions of closed orbits separated by a double homoclinic loop.

Each loop encloses a center at $(\pm\sqrt{\cos(\tau)}, 0)$, around which lies a family of closed orbits. A third family of closed orbits encircles the separatrix. As τ changes, the instantaneous phase portrait changes so that the double homoclinic loop is born, grows to a maximum, shrinks back into the origin, disappears for half a period, and then is born again. This occurs smoothly and periodically.

The nature of the chaotic dynamics exhibited by this system may be observed by numerically integrating eq.(23), see Figs.3 and 4. We see that after each birth of the double homoclinic loop, the numerically integrated trajectory changes from left to right separatrix loop in an unpredictable fashion.

AVERAGING

Since eq.(23) is of the general form (11), we may obtain the averaged equations by using the previously described MACSYMA program. With a solution in the form of eqs.(6),(12)

$$(25) \quad x = r \operatorname{cn}(4K\phi, k)$$

the averaged eqs.(18) are found to be:

$$(26.1) \quad \dot{r} = -\frac{\epsilon}{r} \left[1 - \frac{E}{K} \right] \sin \tau + O(\epsilon^3)$$

$$(26.2) \quad \dot{\phi} = \frac{a}{4K} + O(\epsilon^2)$$

in which $K = K(k)$ and $E = E(k)$ are complete elliptic integrals of the first and second kinds, respectively, and where

$$(27) \quad k^2 = \frac{\rho}{2(\rho - \cos \tau)}, \quad a = \sqrt{\rho - \cos \tau}$$

For convenience we will drop the bar on ϕ and we will define

$$(28) \quad \rho = \frac{-2}{r^2}$$

Rotating Plane Pendulum

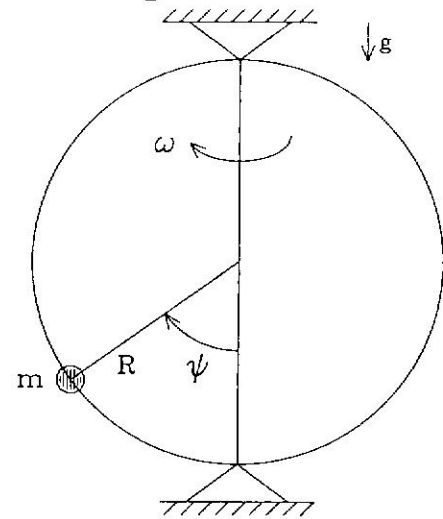


Fig.1 A rotating-plane pendulum. The mass m is constrained to move frictionlessly on a circle of radius R with generalized coordinate ψ . The plane of the circle is constrained to rotate about a vertical axis with given angular velocity $\omega(t)$.

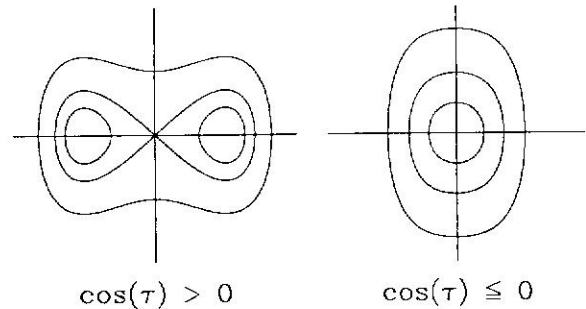


Fig.2 The qualitative nature of the instantaneous phase portrait in the $x-\dot{x}$ plane for system (23) depends on the value of $\cos(\tau)$. For $\cos(\tau) > 0$, a separatrix divides the phase space into three regions; for $\cos(\tau) \leq 0$, no separatrix is present and all orbits encircle the origin.

whereupon eq.(26.1) can be written

$$(29) \quad \frac{d\rho}{d\tau} = -2\epsilon \left[1 - \frac{E}{K}\right] \sin \tau$$

Note that eq.(29) on $\rho(\tau)$ does not depend on φ .

Let us review the significance of ρ and φ . At a particular time τ , ρ determines which instantaneous energy curve a motion is located on, and φ determines where the motion is on that curve. Since, from eq.(29), ρ does not depend on φ , the averaging solution predicts that points which lie on the same instantaneous energy curve at one time, remain together on another instantaneous energy curve at a later time.

Now consider eq.(29). Since from (27), k is a periodic function of τ , and since E and K depend on k , the right hand side of (29) is periodic in τ with period 2π . Thus the phase space of eq.(29) is a

half-cylinder $\mathcal{M} = \{(\rho, \tau) \in \mathbb{R}^+ \times S\}$. From (27) we see that the slope field of (29) is an odd function of τ on \mathcal{M} . Thus the transformation $\tau \rightarrow -\tau$ leaves (29) invariant, and all orbits are symmetric about the line $\tau = 0$. Since (29) has no singularities, there can be no equilibria in the flow on \mathcal{M} . Moreover no orbits can

escape to infinity since for large ρ , $k \sim 1/\sqrt{2}$, $E \sim 1.350644\dots$, $K \sim 1.854075\dots$, so that (29) gives $\frac{d\rho}{d\tau} \sim -0.543053\dots \sin \tau$ and $\rho \sim 0.543053\dots \cos \tau + \rho_0$,

which is a closed orbit. The possibility of limit cycles in the flow on \mathcal{M} is precluded by the Hamiltonian nature of the unperturbed problem. Thus all orbits are closed and we may conclude that for arbitrary initial conditions, $\rho(\tau)$ is an even 2π -periodic function.

This is illustrated in Fig.5, which shows the results of numerical integration of eq.(29). As ρ changes, a given motion moves (slowly, since τ is slow time) from one instantaneous energy curve (24) to another. In order to display the instantaneous position of the separatrix, we substitute (25) and (28) into (24) to obtain

$$(30) \quad \frac{1}{4} \rho (\rho - 2 \cos \tau) = h$$

Since the separatrix passes through the origin $x = \dot{x} = 0$, it corresponds to $h = 0$, from (24). Then (30) gives

$$(31) \quad \text{separatrix: } \rho = 2 \cos \tau > 0$$

We see from Fig.5 that some motions always remain outside the separatrix, while others cross through it, moving between those regions inside the separatrix and those outside the separatrix. See Fig.6. Note that the region of \mathcal{M} lying between $\rho = 0$ (the origin) and $\rho = \cos \tau$ (the instantaneous centers lying inside the homoclinic loops) is unreachable from any physically meaningful initial conditions and should be ignored.

It is important to note that mirror image pairs of instantaneous energy curves which lie inside the separatrix correspond to the same value of ρ . We have seen that points which correspond to the same value of ρ at one time, remain together and correspond to another value of ρ at a later time. Thus in Fig.6, points on the dashed energy curve at B have been split apart into disconnected two sets at C, and find themselves rejoined into a single set again at B'. See Fig.7.

During these motions, the location of a particular point on the known energy curve is determined by the eq.(26.2) on φ . From the fact that $\rho(\tau)$ is an even

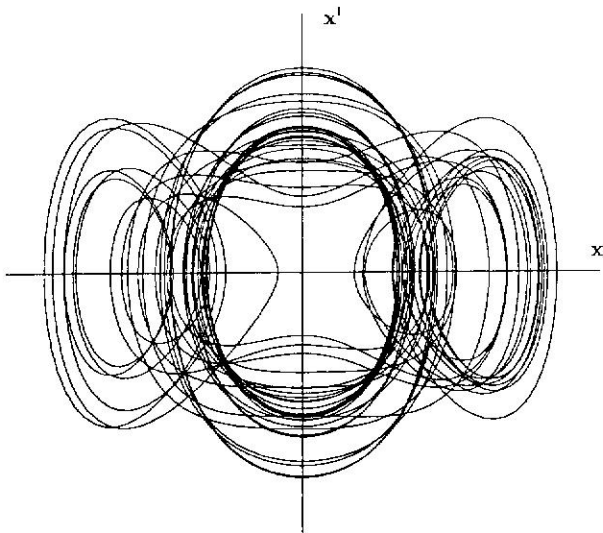


Fig.3. Numerical integration of eq.(23) displayed on the $x-\dot{x}$ plane for $\epsilon = 0.1$. The motion exhibits oscillatory behavior about the origin, to the left of the origin and to the right of the origin.

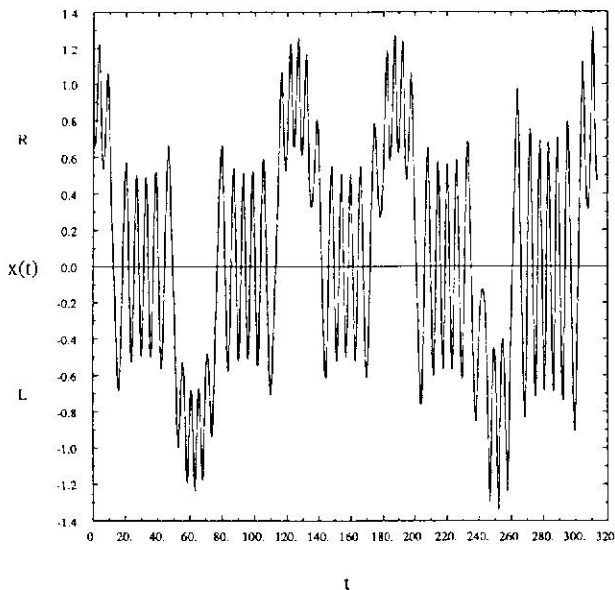


Fig.4. Numerical integration of eq.(23) displayed on the $t-x$ plane for $\epsilon = 0.1$. Oscillations on the right occur for $x > 0$, and on the left for $x < 0$. This motion's symbol sequence is RLRLRL...

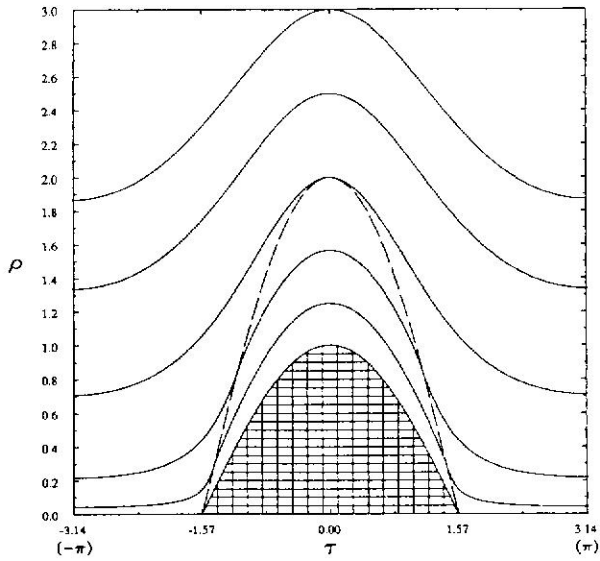


Fig.5 Numerical integration of eq.(29) on the (ρ, τ) half-cylinder phase space \mathcal{M} . The separatrix condition eq.(31) is shown dashed; some orbits cross this curve (corresponding to crossing into or out of a separatrix loop) while others do not. We denote by τ_s the time τ at which an orbit starting at $\tau = 0$ first crosses the separatrix condition curve (for orbits which do cross). The gridded region is unreachable from any physically meaningful initial conditions and should be ignored.

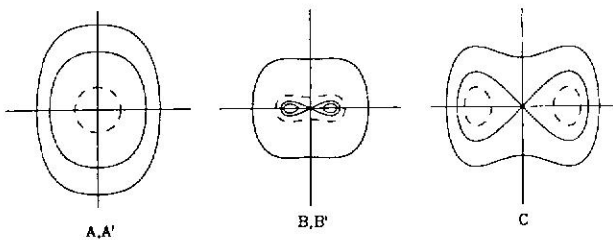
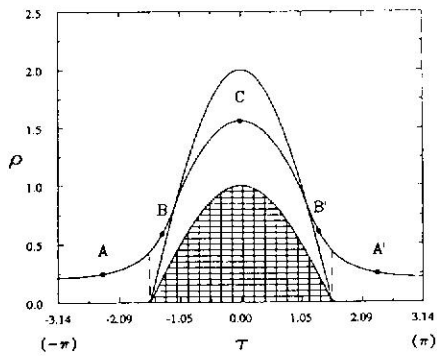


Fig.6 Relation between the flow on the (ρ, τ) half-cylinder phase space \mathcal{M} of Fig.5 and the $x-\dot{x}$ instantaneous phase plane of Fig.2. A trajectory $ABCB'A'$ in \mathcal{M} is shown which intersects the separatrix condition eq.(31), cf. Fig.5. The instantaneous phase portraits in the $x-\dot{x}$ plane for the points A, B, C, A', B' are also shown. The instantaneous energy curves for these points are shown dashed in the phase portraits.

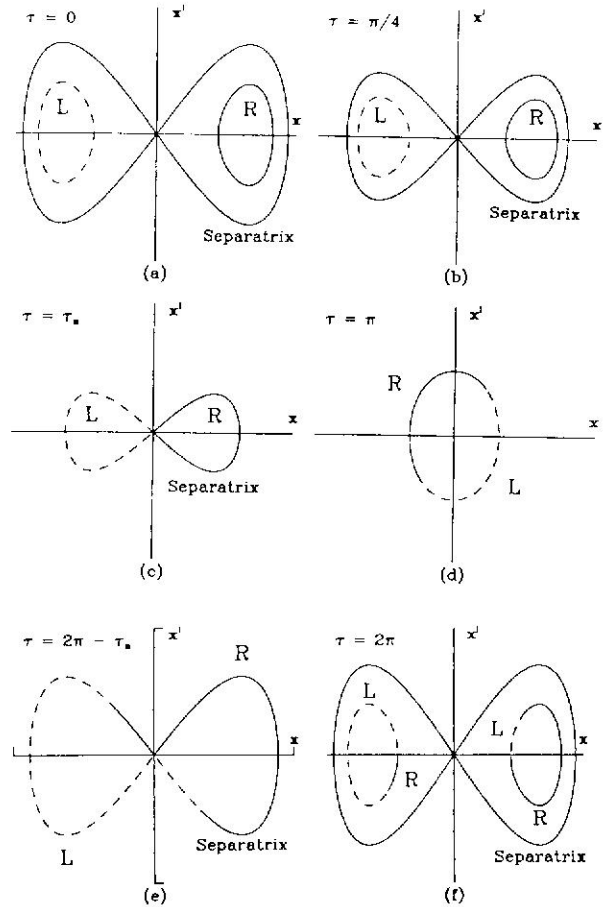


Fig.7 The evolution over one forcing period of the instantaneous energy curves shown in the $x-\dot{x}$ instantaneous phase plane. The points originally on curve L are shown dashed, while those on curve R are shown solid. At $\tau = \tau_s$ the two disconnected sets are joined as the separatrix is crossed. From $\tau = \tau_s$ until $\tau = 2\pi - \tau_s$ the points rotate around the instantaneous energy curve. Then at $\tau = 2\pi - \tau_s$ the separatrix is again crossed and the points again separate into two disconnected sets. The net result is the Poincaré map shown in (f), which consists of a rotation combined with an interchange of two intervals between L and R.

2π -periodic function, it follows that so are $r(\tau)$, $a(\tau)$ and $k(\tau)$. From (26.2), this makes $\frac{d\varphi}{dt}$ an even 2π -periodic function of τ with a nonzero mean. Hence $\varphi(\tau)$ can be written as:

$$(32) \quad \varphi(\tau) = \varphi_{av} \tau + \varphi_{per}(\tau) + \varphi_0$$

where $\varphi_{per}(\tau)$ is an odd 2π -periodic function. Both φ_{av} and $\varphi_{per}(\tau)$ depend on ρ_0 , the initial value of ρ , but are independent of φ_0 .

CHAOS

In order to better understand the dynamics of the averaged system, we consider a Poincare map. We choose a surface of section corresponding to $\tau = 0 \pmod{2\pi}$. Suppose that for a given value of ρ at time $\tau = 0$, there are two disconnected instantaneous energy curves (L = left, R = right), each lying within its own homoclinic loop. The Poincare map maps the set of points $C = L \cup R$ onto itself. By considering the φ flow (32), it is possible to show (Coppola (1989)) that the action of the Poincare map on the set C consists of:

- i) an interchange of two intervals between R and L, and
- ii) a rotation by $2\pi \varphi_{av}$, which is generically

incommensurate with the circumference of the instantaneous energy curve C.

The Poincare map is illustrated in Fig.7, cf. Figs.7a and 7f.

Thus the endpoints Q of the interchanged intervals separate initial conditions which end up on opposite sides of the homoclinic loops after one forcing period. If the Poincare map is iterated, the set of endpoints Q will, in the generic case, continue to grow, and in the limit as $\tau \rightarrow \infty$, this set will be dense in C. Thus the sequence of L's and R's corresponding to a particular point on C will be different from those of neighboring points on C. This implies sensitive dependence on initial conditions, a criterion often used to describe chaos (Guckenheimer and Holmes (1983)).

The Poincare map may be characterized by a one dimensional discontinuous function $F(\Psi)$ which maps the set C onto itself. $F(\Psi)$ is an example of an interval exchange map (Keane (1975)). If we parameterize C by letting Ψ go from -1 to 0 on L and from 0 to 1 on R, then the graph of the function $F(\Psi)$ lies in the square region $[-1,1) \times [-1,1)$ and is displayed in Fig.8a. In the generic case, high iterates of $F(\Psi)$ will contain many points of discontinuity, see Fig.8b.

SUMMARY

We have written a computer algebra (MACSYMA) program which implements the method of averaging to perturb off of strongly nonlinear oscillators (1). The program has been applied to the slowly varying system (23), which describes the dynamics of a rotating-plane pendulum. By analyzing the averaged equations we have been able to obtain a simplified explanation of the kind of chaos which is observed in numerical integrations of eq.(23).

ACKNOWLEDGEMENT

The authors wish to thank Professor Philip Holmes of Cornell University for suggesting the chaotic example of eq.(23).

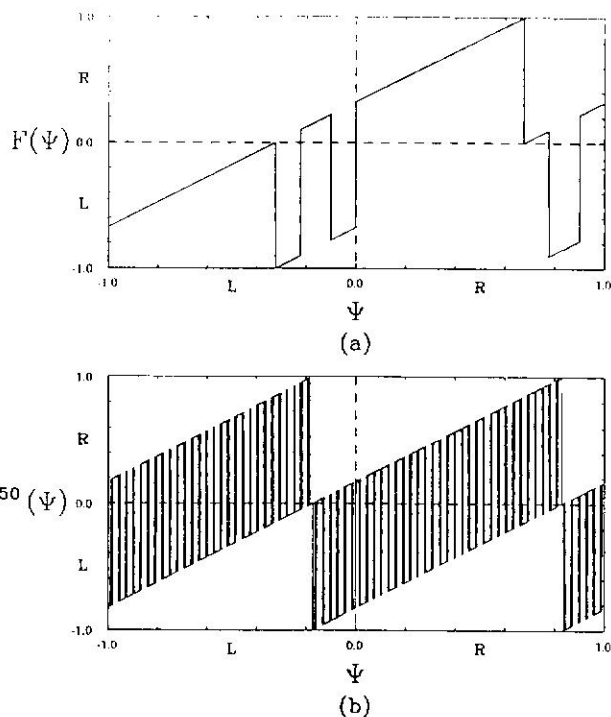


Fig.8 The Poincare map is displayed as the interval exchange map $F(\Psi)$ in (a). Each point on C (where $C = L \cup R$) corresponds to a unique Ψ value which gets mapped under F to another point of C. F is composed of an exchange of an interval between R and L and a translation in Ψ . The 50th iterate of F , which shows many exchanged intervals, is shown in (b).

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