

A Mathematical Study of Resonance in Intact Fruits and Vegetables Using a 3-Media Elastic Sphere Model

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A conceptual framework for the interpretation of non-destructive, resonance studies of the texture of intact fruits and vegetables is developed. The internal mechanical properties of intact specimens (e.g. apples, peaches and cantaloupes) modelled as having 3 mechanically distinct, concentric, spherical regions are mathematically related to the intact resonant frequencies and the associated mode shapes are identified. Also, the inverse problem, that of inferring the shear moduli from frequency measurements, is developed and a computer implementation of the computational procedure is described.

1. Introduction

Objective measures of texture have assumed an increased importance in the growth, harvesting, processing and marketing of fruits and vegetables. If the product quality can be related to the measurable, intact, mechanical properties in a meaningful and reproducible manner, then decisions concerning the harvesting process, probable storage life, etc., may be made more intelligently. A non-destructive procedure for inferring the shear modulus of intact fruits should permit greater insight into the mechanical aspects of growth.

Although a U.S. patent (#2,277,037) was issued in 1942 for a device to measure the transmission of mechanical vibrations through fruits for the purpose of measuring ripeness, only relatively recently has interest developed in the utilization of the technique. The most extensively studied procedure utilizes small cylindrical specimens cut from the fruit or vegetable for vibratory measurements (due primarily to the simplification associated with simple geometries and homogeneous samples). However, to circumvent the rapid deterioration associated with freshly cut tissue, to minimize the errors related to the tolerances for specimen dimensions, and importantly, to make the test non-destructive, the behaviour of *intact* fruits and vegetables has been investigated experimentally.^{1–10} In this paper a theoretical basis for these studies will be developed.

The non-destructive nature of the technique with intact specimens permits repetitive tests with the same fruit (either while growing in the orchard or while in storage) and, thereby, permits the removal of an important source of variability in many experiments. The test procedure consists of making inferences about the internal properties of the fruit based upon the transmission through the fruit of vibratory motion produced at the surface and detected at some other point on the surface.

This testing method has been implemented with at least 3 forms of excitation: (1) sinusoidal;^{1–10} (2) pulse¹¹ and (3) random.¹² Since the behaviour under the second and third conditions can, in principle, be related to the response to a sinusoidal excitation and since this form of excitation has been more widely used in experiments, it will form the basis for discussion here. Abbott *et al.*^{4, 5} have used a tangentially applied sinusoidal signal while Finney^{6, 7} and colleagues have used a radially applied sinusoidal excitation. As the excitation frequency is varied from approximately 20 Hz to more than 4000 Hz, a sensor on the opposite side of the fruit is used to detect the resonance conditions.

The present study develops the concepts required to infer physically significant information about the interior of fruits from a knowledge of the (natural) frequencies associated with the relative maxima of detected motion and the corresponding vibration patterns (mode shapes). Reported acoustic spectra for apples will be examined and related to a normal mode analysis.

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Finally, a computer implemented computational procedure for finding the shear moduli of the skin, flesh and core (or pit) of successively peeled apples (peaches) is described.

2. Review of literature

The acoustic spectra for apples has been reported by Abbott³⁻⁵ and Finney.^{6, 7} Abbott and colleagues also examined the mode shapes associated with the principal resonant frequencies. The vibration at the lowest resonance peak was found to be of a fundamentally different character to that of the second to fifth peaks. Since the nomenclature used previously is more appropriate for cylindrical rather than spherical objects, an alternative designation for the mode shapes (consistent with that used in geophysical studies) will be proposed in this paper. In fact, the analogy between the free vibrations of the earth and of apples is sufficiently fertile as to justify a review of some of the relevant geophysical literature. Reviews by Stoneley,¹³ Bolt¹⁴ and Bullen¹⁵ provide an overview of the geophysical studies. The pioneering works of Lamb¹⁶ (1882) and Love¹⁷ (1911) provided insight into the general behaviour of the free vibrations. Lamb¹⁷ showed that the normal modes of vibration for a homogeneous elastic sphere can be treated in two distinct classes.^{13, 18, 19} The first class results in a dilatation-free (no volume change) motion in which the radial displacement component vanishes everywhere and is referred to as the torsional class. The torsional class is sometimes also called toroidal. All motion in this class is restricted to concentric, spherical surfaces.

The other class of normal mode vibrations is called spheroidal. The motion is, in general, such that the volume is not constant during the oscillation and the displacement vector requires three independent components. The radial component of the curl of the displacement vanishes everywhere.

A notation for identifying the various modes of vibration was introduced by MacDonald & Ness²⁰ in 1961. Let ${}_i T_n^m$ and ${}_i S_n^m$ denote respectively, vibrations for the torsional and spheroidal classes. The surface deformation is characterized by the n and m indices of the associated Legendre function, $P_n^m(\cos \theta)$, $n \geq m$, which appears in the eigenfunction. The subscript n indicates the mode and $n-m$ gives the number of nodal lines parallel to the equator. The symbol i refers to the serial numbering (according to magnitude) of the infinite number of roots of the frequency equation for given n . (The integer m does not appear in the frequency equation.) The lowest or fundamental frequency will be denoted by $i=0$ and the overtones by $i=1, 2, \dots$. For the torsional class, i is also the number of concentric spherical nodal surfaces.

A dimensionless natural frequency s (not to be confused with the notation for identifying the class 2 vibration) is expressed in terms of the angular natural frequency ω , radius a , mass density ρ and shear modulus μ ,

$$s = \omega a (\rho/\mu)^{\frac{1}{2}}. \quad \dots(1)$$

The natural frequency ω for the torsional, but not spheroidal, class depends upon only one of the 2 independent elastic parameters, μ . A few of the infinite number of mode shapes, each of which is associated with a specific natural frequency, will now be described.

2.1. Torsional class

For $n=1$, the rotary vibrations, the fundamental frequency is zero and the mode corresponds to rigid-body oscillation about the z -axis (polar axis). The next higher root corresponds to the interior of the sphere periodically rotating in a direction opposite to that of the outer portion. A single concentric spherical surface at $r/a=0.7796 \dots$ corresponds to a surface of zero displacement (a nodal surface). The third root corresponds to concentric regions oscillating about the z axis, with nodal surfaces at $r/a=0.4792 \dots$ and $r/a=0.8283 \dots$. The next higher overtone will have four such regions and so on.

For $n=2$, particles in each plane of constant z oscillate about the z axis; motions have equal magnitudes but opposite senses at equal distances above and below the equatorial plane and reverse each half cycle. The mode denoted by ${}_0 T_2^2$ is of principal interest in this paper. For 2

great circles through the polar axis the north-south (NS) tangential displacement component is zero; displaced 45° from these great circles the east-west (EW) tangential component is zero, as is also the case along the equator. The maximum NS displacement occurs at 4 points along the equator and the maximum EW component (which is always less than the maximum NS component) occurs at 4 points along both the 45° and the 135° colatitudinal lines. For T_2^2 the NS (θ direction) displacement u_θ , the EW (ϕ direction) displacement u_ϕ , and the vector sum is shown for several ϕ values in Fig. 1.

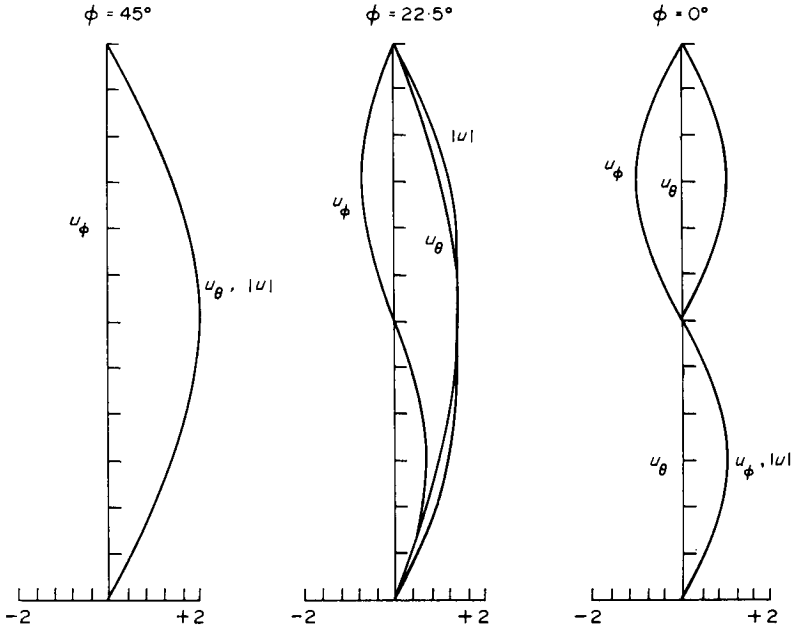


Fig. 1. ${}_0T_2^2$ components of surface displacement. The upper and lower end points of the vertical axes correspond to the N and S poles of the sphere, respectively

2.2 Spheroidal class

For the spheroidal class the volume is not constant during the oscillation. For $n = 0$ the motion is purely radial. In other words, the spherical shape, but not size, is maintained by this “breathing” motion. The $n = 1$ case does not exist since a restraining force would be required to keep the centre of mass at rest (i.e. to prevent translation). The $n = 2$ case is especially important in geodynamics and in this work. It occurs at a lower frequency than for the $n = 0$ (radial) case and is characterized by the sphere’s oscillation between the shape of a prolate and an oblate spheroid. The $n = 3$ case is more complicated and is not necessary for our purposes. (A very detailed discussion of the mode shapes for a thin elastic membrane spherical shell has been presented by Baker.²¹ The configurations are symmetric about the polar axis and are quite similar to those for the sphere. Nonaxisymmetric modes have been considered by Silbiger.²²)

Tables I and II give dimensionless frequency values s (Eqn 1) for some of the torsional and spheroidal modes, respectively; these values will be useful estimates for the 3 media model to be developed below. The increase in frequency from the fundamental to higher overtones (increasing i) is monotonic; for the torsional class the difference between successive overtones approaches π as i increases. The frequency increases for fixed i with increasing values of n , except for the transition from $n = 0$ to $n = 2$ for the spheroidal class.

For a homogeneous, isotropic, elastic, non-gravitating, non-rotating sphere the lowest dimensionless torsional frequencies in ascending order are 2.50, 3.86, 5.09 and 5.76 for ${}_0T_2$, ${}_0T_3$, ${}_0T_4$

TABLE I
Torsional class nondimensional free vibration frequencies of a homogeneous elastic sphere, $s = \omega a(\rho/\mu)^{1/2}$

<i>i</i>	<i>n</i> =1	<i>n</i> =2	<i>n</i> =3	<i>n</i> =4	<i>n</i> =5
0	0.0000	2.5011	3.8647	5.0946	6.2658
1	5.7635	7.1360	8.4449	9.7125	10.9506
2	9.0950	10.5146	11.8818	13.2109	14.5109
3	12.3229	13.7717	15.1754	16.5445	17.8858

and ${}_1T_1$ respectively. On the other hand, for the spheroidal class the order depends upon the value of Poisson's ratio ν . For $\nu=0.3$ the sequence is 2.65, 3.94, 5.00, 5.01 and 5.05 for ${}_0S_2$, ${}_0S_3$, ${}_0S_0$, ${}_1S_2$ and ${}_0S_4$ respectively. Observe that the ${}_0T_2$ mode has the lowest frequency.

Benioff, who identified the observed 57-min period oscillation of the earth following the 1952 Kamchatka earthquake as one of the earth's free vibrations, stimulated renewed interest in theoretical studies. Sato & Matumoto^{23, 24} have studied the torsional oscillations for a 2 media sphere while Gilbert & MacDonald²⁵ applied the Thomson-Haskell matrix method to layered-sphere models of the earth. On the other hand, Alterman, Jarosch & Pekeris^{26, 27} formulated

TABLE II
Spheroidal class nondimensional free vibration frequencies of a homogeneous elastic sphere, $s = \omega a(\rho/\mu)^{1/2}$

<i>i</i>	ν	<i>n</i> =0	<i>n</i> =0	<i>n</i> =3	<i>n</i> =4
0		${}_0S_0$	${}_0S_2$	${}_0S_3$	${}_0S_4$
	0.0	2.9438	2.5921	3.7640	4.7610
	0.1	3.3977	2.6152	3.8360	4.8747
	0.2	4.0180	2.6327	3.8927	4.9689
	0.3	4.9955	2.6462	3.9374	5.0460
	0.4	7.0952	2.6569	3.9731	5.1092
1		${}_1S_0$	${}_1S_2$	${}_1S_3$	${}_1S_4$
	0.0	8.4010	4.1159	5.6443	7.2111
	0.1	8.9700	4.4129	5.9597	7.5475
	0.2	9.8458	4.7171	6.2920	7.8959
	0.3	11.3971	5.0070	6.6086	8.2147
	0.4	15.1225	5.2619	6.8811	8.4745
2		${}_2S_0$	${}_2S_2$	${}_2S_3$	${}_2S_4$
	0.0	13.0190	7.3067	8.8027	10.1642
	0.1	13.8458	7.6340	9.1069	10.4555
	0.2	15.1238	8.0884	9.5024	10.8375
	0.3	17.4004	8.5272	9.8824	11.2250
	0.4	22.9103	8.7591	10.1362	11.5163
3		${}_3S_0$	${}_3S_2$	${}_3S_3$	${}_3S_4$
	0.0	17.5425	9.0494	10.6252	12.1989
	0.1	18.6339	9.1699	10.8406	12.4908
	0.2	20.3230	9.4539	11.3017	13.0631
	0.3	23.3374	10.3759	12.4277	14.1626
	0.4	30.6503	12.0138	13.4255	14.8026
	0.5		12.1607		14.9754

the equations for both classes for radially variable properties (including gravitational attraction) in a manner suitable for direct integration by a digital computer. The 1960 Chilean earthquake provided the basis for important agreement and productive interaction between the theoretical and experimental studies.¹⁴

3. Formulation of a resonance model for fruits and vegetables

Abbott *et al.*³⁻⁵ in an experiment with a pair of stem suspended Rome Beauty apples of different maturity, but otherwise well-matched, measured the lowest 4 resonant peaks when excited by successively varied frequencies between 20 and 4000 Hz. The lowest frequency mode was termed "longitudinal" and the remaining modes were classed "flexural". In view of the aforementioned nomenclature,²⁰ we conjecture that the "longitudinal" vibration corresponds to ${}_0S_2$; and the "flexural" vibrations correspond to ${}_0T_2$, ${}_0T_3$ and ${}_0T_4$, respectively. The same matched apples were then peeled, successively removing portions of the skin, flesh and core and the resonance for the lowest "flexural" frequency (widely denoted as f_2 but now recommended as ${}_0T_2^2$) was noted.

These 2 experiments constitute the principal focus of the work which follows. First, a model to predict the resonant frequencies of both classes will be developed. Then, a procedure for the computation of the shear modulus of the apple skin, flesh and core will be developed. This inverse problem is more direct than is available to the seismologist since we can happily dissect and discard our test specimen.

A 3-media elastic sphere model will be employed in order for the distinct anatomical parts to be considered to have distinct mechanical properties. The assumption of elastic properties appears justified on the basis of the small strains involved. The damping is not expected to produce an appreciable difference between the forced resonant frequency and the undamped free vibration frequency.

The well known Navier's equation for small motions of a homogeneous, isotropic elastic medium under no body forces is

$$(\lambda + 2\mu)\nabla(\nabla \cdot \bar{u}) - \mu \nabla \times \nabla \times \bar{u} = \rho \frac{\partial^2 \bar{u}}{\partial t^2} \quad \dots(2)$$

where λ and μ are Lamé constants, \bar{u} is the displacement vector, and ρ is the mass density. In the case of a viscoelastic medium λ and μ are complex quantities and are frequency-dependent, but for the present analysis they are real, non-negative parameters. If the displacement vector \bar{u} is decomposed²⁸ into a scalar potential ϕ and a vector potential $\bar{\psi}$, then the solution of Eqn (2) will reduce to the study of a scalar and vector wave equation. If

$$\bar{u} = \nabla \Phi + \nabla \times \bar{\psi} \quad \dots(3)$$

where

$$\nabla \cdot \bar{\psi} = 0 \quad \dots(4)$$

then Eqn (2) can be solved by the simultaneous solution of

$$c_c^2 \nabla^2 \Phi = \ddot{\Phi} \quad \dots(5)$$

and

$$-c_s^2 \nabla \times \nabla \times \bar{\psi} = \ddot{\bar{\psi}} \quad \dots(6)$$

where the positive constants are compressional phase velocity

$$c_c = [(\lambda + 2\mu)/\rho]^{\frac{1}{2}} \quad \dots(7)$$

and the shear phase velocity

$$c_s = (\mu/\rho)^{\frac{1}{2}}. \quad \dots(8)$$

Each region of the model will be characterized by an outer radius, a mass density, a shear modulus and Poisson's ratio. The regions will be numbered consecutively, with the innermost being 1. The displacement for each region satisfies Eqn (2). We shall also assume that (a) there is no stress at the outer surface of the fruit; (b) the stress components are continuous across the interfaces between regions; and (c) the displacement components are also continuous across the interfaces. The resonant frequency equations will be developed for both the torsional and spheroidal classes. Notes on a FORTRAN IV implementation of the equations will be given in the Appendix.

3.1. Torsional class natural frequencies for a three-media elastic sphere

The two-layer spherical model of Sato & Matumoto²⁴ will now be generalized to a 3-media sphere. The displacements for the torsional class which satisfy Eqn (1) are expressed in spherical coordinates as

$$u_r = 0 \quad \dots(9)$$

$$u_\theta = \frac{-im}{n(n+1)} \begin{pmatrix} A_1 j_n(kr) \\ A_2 y_n(kr) \end{pmatrix} \frac{1}{\sin \theta} P_n^m(\cos \theta) \exp(im\varphi) \exp(i\omega t) \quad \dots(10)$$

$$u_\phi = \frac{1}{n(n+1)} \begin{pmatrix} A_1 j_n(kr) \\ A_2 y_n(kr) \end{pmatrix} \frac{d}{d\theta} P_n^m(\cos \theta) \exp(im\varphi) \exp(i\omega t) \quad \dots(11)$$

$$\text{where } k = \omega(\rho/\mu)^{\frac{1}{2}}, \quad \dots(12)$$

j_n and y_n are the spherical Bessel functions and P_n^m is the associated Legendre function.

This set of equations applies to each media where ρ and μ in Eqn (12) are, in general, different for each layer. At the outer surface of the deformed outer medium (i.e. at $r=r_3+u_r$) the stresses $\tau_{rr} = \tau_{r\theta} = \tau_{r\phi} = 0$. If only first-order terms are retained

$$\tau_{rr} = \lambda \nabla \cdot \bar{u} + 2\mu \frac{\partial u_r}{\partial r} = 0, r=r_3 \quad \dots(13)$$

$$\tau_{r\theta} = \mu \left[\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right] = 0, r=r_3 \quad \dots(14)$$

$$\tau_{r\phi} = \mu \left[\frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right] = 0, r=r_3. \quad \dots(15)$$

But since this class has zero dilatation ($\nabla \cdot \bar{u} = 0$) and no radial displacement ($u_r = 0$) Eqn (13) is satisfied identically and Eqns (14) and (15) may be simplified to

$$\frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) = 0 \quad \dots(16)$$

and

$$\frac{\partial}{\partial r} \left(\frac{u_\phi}{r} \right) = 0. \quad \dots(17)$$

But when u_θ and u_ϕ from Eqns (10) and (11) are substituted into Eqns (16) and (17) the equations are identical. Therefore only one stress equation need be satisfied at each interface. For example, between media 2 and 3 we require

$$\mu_2 \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right)_2 = \mu_3 \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right)_3, r=r_2. \quad \dots(18)$$

Since the displacement is bounded at the origin, the spherical Bessel functions of the second kind in Eqn (10) may be discarded for medium 1. The natural frequencies may now be computed from the 5 simultaneous, homogeneous, linear equations obtained by substituting the general solution [Eqns (9, 10, 11)] into the 5 conditions (1) traction-free surface at r_3 ; (2) stress continuity at r_2 ; (3) displacement continuity at r_2 ; (4) stress continuity at r_1 ; and (5) displacement continuity at r_1 . The resonant frequencies may be obtained in dimensionless form as the roots ζ of the following determinantal equation where each row has been obtained by substituting Eqn (10) with the appropriate k value into the corresponding 5 conditions above.

$$\begin{vmatrix} \frac{\partial j_n(\zeta)}{\partial \zeta} \frac{1}{\zeta} & \frac{\partial y_n(\zeta)}{\partial \zeta} \frac{1}{\zeta} & 0 & 0 & 0 \\ \frac{\partial j_n(\gamma)}{\partial \gamma} \frac{1}{\gamma} & \frac{\partial y_n(\gamma)}{\partial \gamma} \frac{1}{\gamma} & \rho_{23} \frac{\partial j_n(\eta)}{\partial \eta} \frac{1}{\eta} & \rho_{23} \frac{\partial y_n(\eta)}{\partial \eta} \frac{1}{\eta} & 0 \\ j_n(\gamma) & y_n(\gamma) & j_n(\eta) & y_n(\eta) & 0 \\ 0 & 0 & \frac{\partial j_n(\beta)}{\partial \beta} \frac{1}{\beta} & \frac{\partial y_n(\beta)}{\partial \beta} \frac{1}{\beta} & \rho_{12} \frac{\partial j_n(\xi)}{\partial \xi} \frac{1}{\xi} \\ 0 & 0 & j_n(\beta) & y_n(\beta) & j_n(\xi) \end{vmatrix} = 0 \dots(19)$$

where $\zeta = k_3 r_3, \eta = k_2 r_2, \xi = k_1 r_1, \gamma = k_3 r_2, \beta = k_2 r_1, k_3 = \omega(\rho_3/\mu_3)^{1/2}, k_2 = \omega(\rho_2/\mu_2)^{1/2}, k_1 = \omega(\rho_1/\mu_1)^{1/2}, \rho_{23} = \rho_2/\rho_3, \rho_{12} = \rho_1/\rho_2$. Note that axisymmetric frequencies are the same as for the non-axisymmetric mode shapes with the same n value, since m does not appear in Eqn (19). If the core were hollow (or liquid filled), the displacement condition could be removed and a 4×4 determinant formed by deleting row 5 and column 5. On the other hand, a rigid core would require the deletion of row 4 and column 5. Equation is valid for all $n \geq 1$.

3.2. Spheroidal class natural frequencies for a three-media elastic sphere

The determination of the natural frequencies for the spheroidal class involves greater computational complexity and since the skin for most fruits is relatively thin, we shall treat the outer medium (No. 3) as a thin shell.²⁹

The solution of Eqn (2) for the axisymmetric class 2 vibrations involves the scalar and vector potentials of Eqns (3) and (4). For medium 1 with $\rho_1, \mu_1, \lambda_1, R_1$,

$$\Phi_1 = A P_n(\eta) j_n(A_1 r) \dots(20)$$

$$\psi_1 = B P_n^1(\eta) j_n(A_2 r) = |\bar{\psi}_1| \dots(21)$$

where

$$A_1^2 = \frac{\rho_1 \omega^2}{\lambda_1 + 2\mu_1}, \dots(22)$$

$$A_2^2 = \frac{\rho_1 \omega^2}{\mu_1} \dots(23)$$

and

$$P_n^1(\eta) = (1 - \eta^2)^{1/2} d[P_n(\eta)]/d\eta \dots(24)$$

with

$$\eta = \cos \theta. \dots(25)$$

For medium 2 with $\rho_2, \mu_2, \lambda_2, R_1$ and R_2

$$\Phi_2 = P_n(\eta)[C j_n(A_3 r) + D y_n(A_3 r)] \dots(26)$$

$$\psi_2 = P_n^1(\eta)[E j_n(A_4 r) + G y_n(A_4 r)] = |\bar{\psi}_2| \dots(27)$$

where

$$A_3^2 = \frac{\rho_2 \omega^2}{\lambda_2 + 2\mu_2} \quad \dots(28)$$

and

$$A_4^2 = \frac{\rho_2 \omega^2}{\mu_2} \quad \dots(29)$$

For medium 3 with parameters ρ_3, ν_3, μ_3 and R_2 and displacements U (tangential, positive in a south-to-north direction) and W (radially outward), the equations for a thin shell of thickness h having negligible bending moments are

$$(1-\eta^2)^{\pm} [(1-\eta^2)^{\pm} U_3]'' + (1-\nu_3) U_3 + (1-\eta^2)^{\pm} (1+\nu_3) W'_3 + \frac{(1-\nu_3) R_2^2 \rho_3 \omega^2}{2\mu_3} U_3 = \frac{(1-\nu_3) R_2^2}{2h\mu_3} (\tau_{r\eta})_3 \Big|_{r=R_2} \quad \dots(30)$$

and

$$[(1-\eta^2)^{\pm} U_3]' + 2W_3 - \frac{R_2^2 \rho_3 \omega^2}{2\mu_2} \left(\frac{1-\nu_3}{1+\nu_3} \right) W_3 = \frac{-(1-\nu_3) R_2^2}{2h\mu_3 (1+\nu_3)} (\tau_{rr})_3 \Big|_{r=R_2} \quad \dots(31)$$

where $' \equiv \frac{d}{d\eta}$.

Eqns (30) and (31) represent the extensional or membrane theory of thin shells. The shell stresses are assumed to be uniform through the shell thickness (stresses due to bending are neglected). This theory³⁰ has been shown to be accurate for very thin shells which are completely closed. The boundary conditions at $r=R_1$ are

$$U_1 = U_2 \quad \dots(32)$$

$$W_1 = W_2 \quad \dots(33)$$

$$\tau_{rr1} = \tau_{rr2} \quad \dots(34)$$

$$\tau_{r\eta 1} = \tau_{r\eta 2} \quad \dots(35)$$

and at $r=R_2$

$$U_2 = U_3 \quad \dots(36)$$

$$W_2 = W_3 \quad \dots(37)$$

where

$$\tau_{rr} = -\frac{\rho \omega^2 \lambda}{\lambda + 2\mu} \Phi + 2\mu \frac{\partial W}{\partial r} \quad \dots(38)$$

$$\tau_{r\eta} = \mu \left[\frac{(1-\eta^2)^{\pm}}{r} \frac{\partial W}{\partial \eta} + \frac{\partial U}{\partial r} - \frac{U}{r} \right] \quad \dots(39)$$

$$U = \frac{(1-\eta^2)^{\pm}}{r} \frac{\partial \Phi}{\partial r} + \frac{\partial \psi}{\partial r} + \frac{\psi}{r} \quad \dots(40)$$

$$W = \frac{\partial \Phi}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \eta} [(1-\eta^2)^{\pm} \psi] \quad \dots(41)$$

Substituting Eqns (38)–(41) into the 6 boundary conditions (32)–(37) constitutes a solution of the problem; the solution consists of 6 linear homogeneous algebraic equations on the unknowns A, B, C, D, E, G . For non-trivial solutions the determinant must vanish. The natural frequencies are the roots of the equation formed by expanding this 6×6 determinant. Let

$$j_n(\Lambda_1 R_1) = M_1 \quad \dots(42)$$

$$j_n(\Lambda_2 R_1) = M_2 \quad \dots(43)$$

$$j_n(\Lambda_3 R_1) = M_3 \quad \dots(44)$$

$$j_n(\Lambda_4 R_1) = M_4 \quad \dots(45)$$

$$j_n(\Lambda_3 R_2) = M_5 \quad \dots(46)$$

$$j_n(\Lambda_4 R_2) = M_6 \quad \dots(47)$$

$$y_n(\Lambda_3 R_1) = M_7 \quad \dots(48)$$

$$y_n(\Lambda_4 R_1) = M_8 \quad \dots(49)$$

$$y_n(\Lambda_3 R_2) = M_9 \quad \dots(50)$$

$$y_n(\Lambda_4 R_2) = M_{10} \quad \dots(51)$$

and let primes on M denote $d(\quad)/dR$. The elements of the 6×6 determinant are as follows:

$$S(1,1) = \{M_1/(\Lambda_3 R_1)\} \quad \dots(52)$$

$$S(1,2) = \{(M_2'/\Lambda_3) + M_2/(\Lambda_3 R_1)\} \quad \dots(53)$$

$$S(1,3) = \{-M_3/(\Lambda_3 R_1)\} \quad \dots(54)$$

$$S(1,4) = \{-M_7/(\Lambda_3 R_1)\} \quad \dots(55)$$

$$S(1,5) = \{(-M_4'/\Lambda_3) - M_4/(\Lambda_3 R_1)\} \quad \dots(56)$$

$$S(1,6) = \{-M_8'/\Lambda_3 - M_8/(\Lambda_3 R_1)\} \quad \dots(57)$$

$$S(2,1) = \{M_1'/\Lambda_3\} \quad \dots(58)$$

$$S(2,2) = \{n(n+1)M_2/(\Lambda_3 R_1)\} \quad \dots(59)$$

$$S(2,3) = \{-M_3'/\Lambda_3\} \quad \dots(60)$$

$$S(2,4) = \{-M_7'/\Lambda_3\} \quad \dots(61)$$

$$S(2,5) = \{-n(n+1)M_4/(\Lambda_3 R_1)\} \quad \dots(62)$$

$$S(2,6) = \{-n(n+1)M_8/(\Lambda_3 R_1)\}. \quad \dots(63)$$

Let $H_1 = (\Lambda_1/\Lambda_3)^2 v_1 / (1 - 2v_1)$

and $H_2 = v_2 / (1 - 2v_2)$

then

$$S(3,1) = \{-H_1 M_1 + M_1''/\Lambda_3^2\} \quad \dots(64)$$

$$S(3,2) = \{-n(n+1)[M_2/(\Lambda_3 R_1)^2 - M_2'/(\Lambda_3^2 R_1)]\} \quad \dots(65)$$

$$S(3,3) = \{M_3 H_2 - M_3''/\Lambda_3^2\} \quad \dots(66)$$

$$S(3,4) = \{H_2 M_7 - M_7''/\Lambda_3^2\} \quad \dots(67)$$

$$S(3,5) = \{n(n+1)[M_4/(\Lambda_3 R_1)^2 - M_4'/(\Lambda_3^2 R_1)]\} \quad \dots(68)$$

$$S(3,6) = \{n(n+1)[M_8/(\Lambda_3 R_1)^2 - M_8'/(\Lambda_3^2 R_1)]\}. \quad \dots(69)$$

Let $H_3 = \mu_1/\mu_2$

then

$$S(4,1) = \{-(2H_3[M_1/(\Lambda_3 R_1)^2 - M_1' / (\Lambda_3^2 R_1)])\} \dots(70)$$

$$S(4,2) = \{H_3[M_2'' / \Lambda_3^2 - 2M_2 / (\Lambda_3 R_1)^2 + n(n+1)M_2 / (\Lambda_3 R_1)^2]\} \dots(71)$$

$$S(4,3) = \{2[M_3 / (\Lambda_3 R_1)^2 - M_3' / (\Lambda_3^2 R_1)]\} \dots(72)$$

$$S(4,4) = \{2[M_7 / (\Lambda_3 R_1)^2 - M_7' / (\Lambda_3^2 R_1)]\} \dots(73)$$

$$S(4,5) = \{-M_4'' / \Lambda_3^2 - [M_4 / (\Lambda_3^2 R_1)^2][n(n+1) - 2]\} \dots(74)$$

$$S(4,6) = \{-M_8'' / \Lambda_3^2 + [M_8 / (\Lambda_3 R_1)^2][n(n+1) - 2]\}. \dots(75)$$

Let $r_{23} = r_2/r_3$, $\mu_{23} = \mu_2/\mu_3$, and $\zeta_{23} = \rho_2/\rho_3$, then let

$$H_4 = \{2 \cdot 0 - [(\Lambda_4 R_2)^2][\mu_{23}(1 - \nu_3)] / [2\rho_{23}(1 + \nu_3)]\}$$

$$H_5 = \{[\Lambda_3 R_2 \mu_{23} r_{23}(1 - \nu_3)] / [(1 - r_{23})(1 + \nu_3)]\}$$

$$S(5,1) = 0 \dots(76)$$

$$S(5,2) = 0 \dots(77)$$

$$S(5,3) = \{-n(n+1)M_5 / [\Lambda_3 R_2] - H_2 H_5 M_5 + H_4 M_5' / \Lambda_3 + H_5 M_5'' / \Lambda_3^2\} \dots(78)$$

$$S(5,4) = \{-n(n+1)M_9 / [\Lambda_3 R_2] - H_2 H_5 M_9 + H_4 M_9' / \Lambda_3 + H_5 M_9'' / \Lambda_3^2\} \dots(79)$$

$$S(5,5) = \{[n(n+1)M_6' / \Lambda_3][-1 + H_5 / (\Lambda_3 R_2)] + [n(n+1)M_6 / (\Lambda_3 R_2)][-1 + H_4 - H_5 / (\Lambda_3 R_2)]\} \dots(80)$$

$$S(5,6) = \{[n(n+1)M_{10}' / \Lambda_3][-1 + H_5 / (\Lambda_3 R_2)] + [n(n+1)M_{10} / (\Lambda_3 R_2)][-1 + H_4 - H_5 / (\Lambda_3 R_2)]\}. \dots(81)$$

Finally let,

$$H_6 = [\Lambda_3 R_2 \mu_{23} r_{23}(1 - \nu_3)] / [2(1 - r_{23})]$$

$$H_7 = (1 - \nu_3)[1 + (\Lambda_4 R_2)^2 \mu_{23} / 2\rho_{23}]$$

$$H_8 = (1 + \nu_3) - H_6 / (\Lambda_3 R_2)$$

$$S(6,1) = 0 \dots(82)$$

$$S(6,2) = 0 \dots(83)$$

$$S(6,3) = \{-n(n+1)M_5 / (\Lambda_3 E_2) + H_7 M_5 / (\Lambda_3 R_2) + 2H_6 M_5 / (\Lambda_3 R_2)^2 + [H_8 - H_6 / (\Lambda_3 R_2)][M_5' / \Lambda_3]\} \dots(84)$$

$$S(6,4) = \{[-n(n+1)M_9 / (\Lambda_3 R_2) + H_7 M_9 / (\Lambda_3 R_2) + 2H_6 M_9 / (\Lambda_3 R_2)^2 + [H_8 - H_6 / (\Lambda_3 R_2)][M_9' / \Lambda_3]\} \dots(85)$$

$$S(6,5) = \{[-n(n+1)M_6 / (\Lambda_3 R_2)][1 - H_8] + H_7 M_6 / (\Lambda_3 R_2) + 2H_6 M_6 / (\Lambda_3 R_2)^2 + [-n(n+1) + H_7][M_6' / \Lambda_3] - H_6 M_6'' / \Lambda_3^2\} \dots(86)$$

$$S(6,6) = \{[-n(n+1)M_{10} / (\Lambda_3 R_2)][1 - H_8] + H_7 M_{10} / (\Lambda_3 R_2) + [-n(n+1) + H_7][M_{10}' / \Lambda_3] - H_6 M_{10}'' / \Lambda_3^2\}. \dots(87)$$

The derivatives of M (i.e. M' and M'') are obtained from

$$M' = \Lambda \frac{d}{d(\Lambda R)} M(\Lambda R) = \Lambda \dot{M} \dots(88)$$

where $\dot{M} = dM/d(\Lambda R)$,

where M corresponds to j or y as given earlier, and where

$$\dot{j}_n = (n/\Lambda R)j_n - j_{n+1} \quad \dots(89)$$

and likewise for y_n . The Bessel ordinary differential equation gives the other needed relation

$$M'' = -\frac{2}{R}M' - \left[\Lambda^2 - \frac{n(n+1)}{R^2} \right] M. \quad \dots(90)$$

For the radial mode ($n=0$) vibrations the 6×6 determinant must be reduced to a 3×3 to avoid extraneous roots since $dP_0/d\eta = 0$ and $\psi \equiv 0$. The $n=1$ case is meaningless as indicated earlier. The natural frequencies may be found as the roots $\Lambda_4 R_2$ of the 6×6 determinant set to zero. For a hollow core with $n \geq 2$ delete rows 1 and 2 and columns 1 and 2; for a rigid core delete rows 3 and 4 and columns 1 and 2. For the radial mode ($n=0$) with a hollow core delete rows 1, 2, 4, 6 and columns 1, 2, 5, 6; for the radial mode with a rigid core delete rows 1, 3, 4, 6 and columns 1, 2, 5, 6.

4. Applications

With the above equations programed (as described in the Appendix) we will now apply the model for 2 interesting checks on the theory. A computational procedure for finding the shear moduli of the apple components will be described and a procedure for computing the acoustic spectra outlined. Consider the (direct) acoustic spectra problem first.

An iterative (interval bisection) technique is used to find the values of the desired non-dimensional frequency roots of the determinants which are set to zero—Eqn (19) for the torsional class and the 6×6 array given by Eqns (52)—(87) for the spheroidal case. A suitable set of dimensionless input parameters was chosen as r_1/r_3 , r_2/r_3 , ρ_1/ρ_3 , ρ_2/ρ_3 , μ_1/μ_3 and μ_2/μ_3 . For the spheroidal calculations the Poisson ratio is required for each of the media. Only ratios are required for input and the output is obtained in the dimensionless form of Eqn (1).

For the inverse problem the shear modulus ratios are the unknown parameters but the dimensional frequency is measured. The radii and the density ratios do not vary greatly within a particular variety and may be obtained by relatively simple measurements. With ${}_0T_2$ frequency measurements for the intact fruit as well as for the successively peeled cases as described by Abbott *et al.*, the shear moduli may be computed. The core, treated as a homogeneous media, is computed explicitly using the s value shown in Table I. Next, the torsional frequency equation, which requires an iterative solution, is used as a subprogram to satisfy the “dimensionless frequency” or rather a dimensionless shear modulus for another iteration scheme on the μ_1/μ_2 ratio. Since μ_1 has been computed, the flesh shear modulus is then known. The procedure is repeated to obtain the skin shear modulus since μ_1 and μ_2 are already known.

4.1. Shear moduli calculations

Using data from the literature⁴ for successively peeled apples, the radial variation of the shear modulus was computed. Fig. 2 suggests that the assumption of minimal radial dependence of the flesh shear modulus was appropriate. The computed shear modulus for the skin is 5 to 10 times that of the flesh, as is clear from the figure and from Table III. Fig. 3 shows the same data for the core and flesh plotted on a different scale. The York Imperial, Stayman and Golden Delicious apples have an *independence* of G with radial distance from the core. The large Rome Beauty apple shows a low shear modulus which increases slightly towards the core. All 4 curves show that the core shear modulus increases and then decreases towards the centroid as might be expected due to the presence of seeds and the seed cavity. Table III shows calculated values for several apple varieties. The computed shear moduli for the flesh appear to be in reasonable³¹⁻³⁴ agreement with results from resonance studies of cylindrical sections and these values

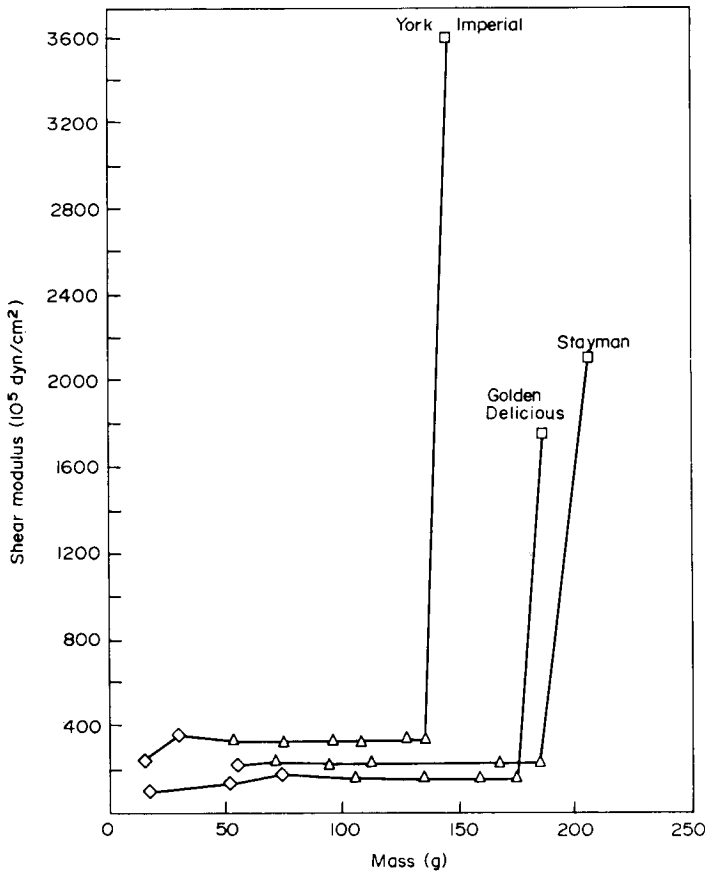


Fig. 2. Shear moduli for successively peeled specimens (original data from Abbott et al.⁴ Fig. 15). □, Skin; △, flesh; ◇, core.

are in the expected relationship to each other. At least one reference³⁵ gives values of Young's modulus for apples (when converted to shear moduli using realistic values of Poisson's ratio^{11, 36}) which are slightly higher than our predictions. Finally the 'well matched' "red" and "green" Rome Beauty apples appear to have slightly different resonant frequencies as much a result of differences in mass as in texture changes. (The calculations are based on graphical rather than original data.)

4.2. Acoustic spectra

The 16 lowest computed free vibration frequencies for the pair of "well-matched"⁴ Rome Beauty apples are listed in Table IV. The parameters used are those obtained from the successive peeling experiment calculations, except for Poisson's ratio which is assumed to be 0.3. Recall that the parameters were obtained by requiring frequency agreement for the ${}_0T_2$ mode. The ordering of the mode shapes is consistent with that reported,⁴ including the appearance of ${}_0S_2$ at a lower frequency than ${}_0T_2$ —in contrast with the result for a homogeneous model. The torsional class modes show reasonable agreement between theory and experiment. On the other hand, the ${}_0S_2$ frequency exhibits a lower correlation for the following reasons: (1) the shear modulus for the core was obtained from torsional data. Since the torsional class involves, theoretically at least, no radial motion, the inner portions of the core contribute less to the results than should be expected for spheroidal vibrations. The seed cavity, then, may have a

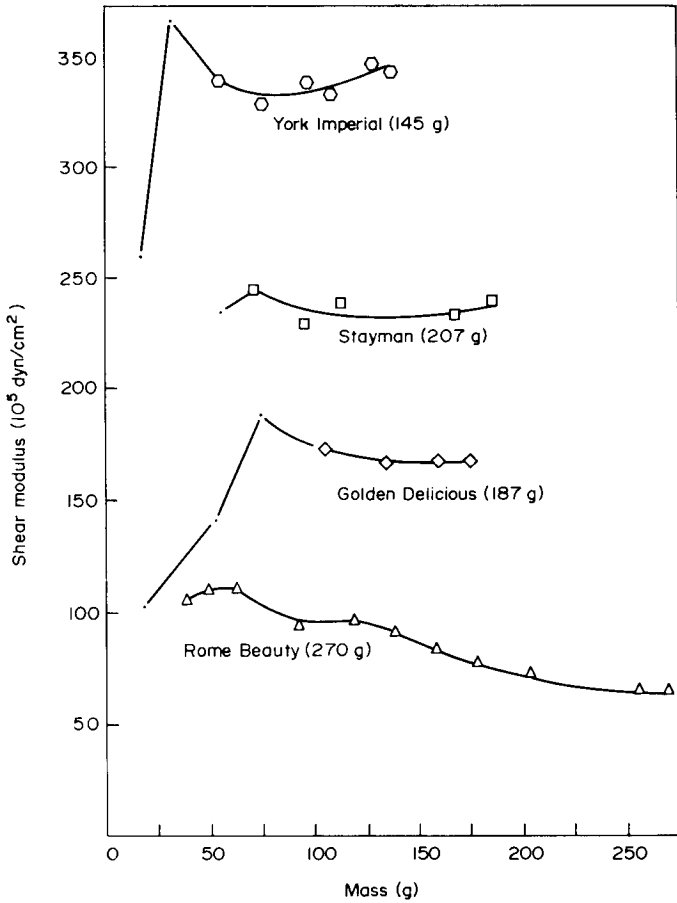


Fig. 3. Radial variation of core and flesh shear moduli for four specimens (original data from Abbott et al.⁴ Figs 13 and 15)

larger influence upon the frequency. A decrease in apparent core shear modulus from 185×10^5 to 20×10^5 dyn/cm 2 for the core would reduce the ${}_0S_2$ frequency to approximately 300 Hz without an appreciable decrease in the torsional frequencies. Finney's data^{6, 7} indicate that the lowest frequency for apples is in the 300 to 400 Hz range. (2) Abbott used an excitation method which enhanced non-radial motion while Finney used an excitation method which did enhance radial motion. Therefore, the Finney method may well be better suited for spheroidal measurements. (3) Finally, the inherent apple geometry, with dimples and with vascular tissue, may be the source of greater variation in the case of spheroidal vibrations.

Several intermediate modes appear within the measured range. It is possible that a further search will disclose some of these. The ordering of the modes by frequency is quite similar for the 2 apples. The ${}_0S_0$ mode is especially sensitive to Poisson's ratio, apparently due to the volume changes associated with this mode. For changes of the Poisson ratio from 0.25 to 0.35 a relatively small change occurs in the spheroidal class frequencies and, as indicated above, no change occurs in the torsional class frequencies.

5. Summary

This mathematical study is intended as a basis for organizing the results from experimental studies and as a guide for further study. The analogy between earthquakes and "applequakes"

TABLE III
Elastic structure of selected apples*

Identification	Mass (g)	Radii (cm)			Shear Moduli (10 ⁵ dyn/cm ²)			Conversion ratio	Dimensionless ratios**			
		Core R ₁	Flesh R ₂	Skin R ₃	Core μ ₁	Flesh μ ₂	Skin μ ₃		R _c /R _s R13	R _f /R _s R23	G _c /G _f MU12	G _s /G _f MU32
Stayman ¹	200	—	3.67	3.82	—	156	1169	1.39	0.48	0.96	—	7.51
Golden Delicious ²	187	2.78	3.70	3.76	189	168	1762	1.58	0.74	0.98	1.12	10.48
Golden Delicious ¹	192	—	3.63	3.77	—	174	902	1.66	0.48	0.96	—	5.18
McIntosh ¹	217	—	3.76	3.93	—	234	1740	1.36	0.48	0.96	—	7.42
Stayman ²	207	2.52	3.76	3.87	235	240	2119	1.45	0.65	0.97	0.98	8.82
Rome Beauty, ³ "Green"	128	2.26	3.19	3.30	222	248	2246	1.36	0.69	0.97	0.90	9.06
Rome Beauty, ³ "Red"	139	2.11	3.30	3.39	185	259	2269	1.52	0.62	0.97	0.71	8.76
York Imperial ²	145	2.08	3.40	3.45	366	343	3605	1.60	0.60	0.98	1.07	10.50
York Imperial ²	156	—	3.42	3.53	—	345	2460	1.54	0.48	0.97	—	7.13
York Imperial ¹	172	—	3.52	3.64	—	394	2491	1.58	0.48	0.97	—	6.32
Red Delicious ¹	238	—	3.97	4.07	—	403	2927	1.66	0.49	0.98	—	7.27

*Calculations based upon data by Abbott *et al.*⁴ The computed values are intended to demonstrate the method and are *not* presented as being statistically representative. Assumed a core and flesh density of 0.83 g/cm³ and skin density of 1.05.

¹ Abbott *et al.* Sonic techniques for measuring texture of fruits and vegetables (1968, Fig. 14).

² *ibid.* (Fig. 15).

³ *ibid.* (Fig. 12).

$$** \frac{R_{core}}{R_{skin}} \frac{R_{flesh}}{R_{skin}} \frac{G_{core}}{G_{flesh}} \frac{G_{skin}}{G_{flesh}}$$

TABLE IV
Acoustic spectra for the matched Rome Beauty apples

Mode	Rome Beauty "Red" NU=0.3 Frequency, (Hz)		Mode	Rome Beauty "Green" NU=0.3 Frequency (Hz)	
	Calculated	Measured		Calculated	Measured
₀ S ₂	680	75	₀ S ₂	723	78
₀ T ₂	823	821	₀ T ₂	877	875
₀ S ₃	1016	1372	₀ S ₃	1063	1478
₀ T ₃	1285		₀ S ₄	1359	
₀ S ₄	1318	1993	₀ T ₃	1363	2107
₀ S ₀	1354		₀ S ₀	1423	
₀ T ₁	1382	1993	₀ T ₁	1460	2107
₁ S ₂	1485		₁ S ₂	1606	
₀ S ₅	1605	1993	₀ S ₅	1635	2107
₀ T ₄	1700		₀ T ₄	1793	
₁ T ₂	1750	1993	₁ T ₂	1851	2107
₀ S ₆	1880		₀ S ₆	1900	
₁ S ₃	1918	1993	₁ S ₃	2064	2107
₀ T ₅	2090		₀ T ₅	2189	
₁ T ₃	2113	1993	₁ T ₃	2239	2107
₁ T ₁	2151		₁ T ₁	2305	

Data taken from Abbott *et al.*⁴ Sonic techniques for measuring texture of fruits and vegetables (1968, Fig. 8).

was developed. The 2 classes of vibrations were described and the notational scheme reviewed. A 3-media elastic sphere model was proposed and the frequency equations for both classes derived. The mode shapes described by Abbott *et al.* were renamed. The acoustic spectra for some apples were computed. Using data from Abbott's successive peeling experiments, the shear modulus for the core, flesh and skin were computed. Finally, some of the main features of the computer programs were described.

In a sequel³⁸ to this paper numerical studies of parameter sensitivities will be described. A non-destructive procedure for computing the flesh shear modulus will be developed and further comments upon experimental procedures will be presented.

6. Conclusions

The 3 media elastic sphere model satisfactorily describes the resonant behaviour of apples which are at or near maturity. The modes associated with the lowest appearing frequencies as described by Abbott *et al.* are ${}_0S_2$, ${}_0T_2^2$, ${}_0T_3^3$, ${}_0T_4^4$. Further experimental study should reveal additional resonant peaks within the frequency range already studied.

The successive peeling experiment was shown to provide the basis for a pure shear calculation of the shear moduli for core, flesh and skin. The skin shear modulus for mature apples appears to be on the order of 5 to 10 times that of the flesh.

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7. Appendix: Computer programs

The FORTRAN IV implementation of the algorithms described will now be highlighted. Copies of the program have been submitted to SHARE⁴² for general distribution.

7.1. Acoustic spectra program

Since many of the required numerical procedures apply to both the torsional and spheroidal classes, both equations are solved by the same program. In general, the frequency equations

appear as 5×5 and 6×6 determinants whose elements involve the spherical Bessel functions. Since only lower order functions are of interest ($n \leq 10$), the ascending recursion equation for the generation of the functions was found to be adequate.³⁹ Finding the roots of the determinant required repetitive evaluation of the determinants; this was accomplished using Chio's pivotal condensation method.⁴⁰ The iterative root search was efficiently accomplished using the bisection method.⁴¹ The search for an interval containing a root was continued until the product of the determinant values at 2 consecutive points became negative. The determinant was evaluated then at the interval mid-point and the interval containing the root was selected for further refinement until the prescribed tolerance was obtained or until the convergence failed. The search for the nondimensional frequency, given by Eqn (1), could be for media 3 subscripts, media 2 or any combination.

7.2. *Shear modulus program*

The iterative shear modulus program uses the acoustic spectra program as a subroutine. The experimental data for the ${}_0T_2^2$ mode resonance from a successively peeled fruit was used as input. After the shear modulus for the core is explicitly obtained using Eqn (1), the shear moduli for the flesh and skin are iteratively obtained. Estimates for the ratio μ_1/μ_2 are used by a renamed, duplicate root bisection subroutine (since FORTRAN does not permit recursive calls to a subroutine) to compute the nondimensional frequency for the flesh. The nondimensional frequencies obtained from the acoustic spectra program and from Eqn (1) are compared. Iteration on μ_1/μ_2 continues until the prescribed tolerance is achieved. Since both μ_1 and μ_1/μ_2 are then known, μ_2 is easily computed. The same procedure is then used to obtain μ_3 .