Stability of the Triangular Points in the Elliptic Restricted Problem of Three Bodies

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The two variable expansion method is used to study the stability of infinitesimal motions about the triangular libration points in the elliptic restricted problem of three bodies. This perturbation technique entails replacing the independent variable (here f, the true anomaly of the smaller primary) with two new independent variables. The results of the study are analytical expressions for the transition curves bounding regions of stability in the $\mu - e$ plane, accurate to $0(e^3)$. For small e, these expressions are seen to compare favorably with the numerical analysis of Danby.

1. Introduction

In the elliptic restricted problem of three bodies, the primaries move in elliptical orbits about their common center of mass, uninfluenced by the third body, which is assumed to have negligible mass. If an equilateral triangle is imagined to move in the plane of motion of the primaries, such that two of its vertices are attached to each of the primaries, respectively, then the third vertex will be an equilibrium point for the third body. The stability of such a triangular equilibrium point is the subject of this paper.

In 1964, Danby[1] investigated the stability of infinitesimal motions about the triangular points in the elliptic restricted problem by using a digital computer numerical scheme based on Floquet theory. His results were given in the form of regions of stability in the $\mu - e$ plane (where μ is ratio of the mass of the smaller primary to the sum of the masses of the primaries, and e is the eccentricity of the orbit of the primaries). See Fig. 1, where μ_a is the limiting value of μ for stable motions in the circular restricted problem and where μ_b is the value of μ such that one of the periods of motion about the triangular points is exactly twice the period of the primaries, in the circular problem. It is desirable to confirm this numerical analysis with an analytic procedure.

In this paper, a two variable expansion method[2,3,6] for treating singular perturbation problems is used to obtain algebraic expressions for the transition curves to $0(e^3)$. In a previous paper,[4] it was shown that the slopes of the two transition curves which intersect the μ axis at $\mu = \mu_b$ differ only by sign at $e = 0$, i.e., there is local symmetry about $\mu = \mu_b$ in the $\mu - e$ plane. In this paper, it is shown that the curvatures of these two curves at $e = 0$ are equal. In addition, the transition curve which intersects the μ axis at $\mu = \mu_a$ is shown to be perpendicular to the μ axis, and an expression for the curvature at $e = 0$ is found.

2. First Variational Equations

The equations of infinitesimal motions of the third body about the triangular liberation points as derived by Szebehely (Ref. 5, p. 598) are

$$u'' - 2v' = g(e,f)[\Omega_{xx}{}^0 u + \Omega_{xy}{}^0 v] \qquad (1a)$$

$$v'' + 2u' = g(e,f)[\Omega_{xy}{}^0 u + \Omega_{yy}{}^0 v] \qquad (1b)$$

where u, v are the pulsating dimensionless coordinates of the third body relative to the triangular point L_4, f is the true anomaly of the smaller primary

$$g(e,f) = 1/(1 + e\cos f) \qquad (2)$$

$$\Omega_{xx}{}^0 = \tfrac{3}{4} \qquad (3)$$

$$\Omega_{xy}{}^0 = 3(3)^{1/2}(\mu - \tfrac{1}{2})/2 \qquad (4)$$

$$\Omega_{yy}{}^0 = \tfrac{9}{4} \qquad (5)$$

and where primes denote differentiation with respect to f.

A principal system of coordinates may be derived, after Szebehely (Ref. 5, p. 254), by setting

$$u = x\cos\alpha - y\sin\alpha \qquad (6a)$$

$$v = x\sin\alpha + y\cos\alpha \qquad (6b)$$

where

$$\tan 2\alpha = (3)^{1/2}(1 - 2\mu) \qquad (7)$$

Then the equations of motion become

$$x'' - 2y' - g(e,f)f_2 x = 0 \qquad (8a)$$

$$y'' + 2x' - g(e,f)f_1 y = 0 \qquad (8b)$$

where

$$f_{1,2} = \tfrac{3}{2}\{1 \pm [1 - 3\mu(1 - \mu)]^{1/2}\} \qquad (9)$$

3. Perturbation Scheme

It was shown by Alfriend and Rand[4] that for an infinitesimal eccentricity of the orbit of the primaries, infinitesimal motions about the triangular points are stable for any value of $0 < \mu < \mu_a$ except for $\mu = \mu_b$. Therefore, for $0 < \mu < \mu_a$ the transition curves intersect the μ axis at $\mu = \mu_b$. For zero eccentricity and $\mu \geqslant \mu_a$ the motion is unstable, hence $\mu = \mu_a$ is a transition point. The remainder of this paper is concerned with solutions of the equations of motions (8) for $\mu \approx \mu_b$ and $\mu \approx \mu_a$.

The two variable expansion method[2,3,6] for treating singular perturbation problems entails replacing the independent variable f by two new independent variables ξ and η,

$$\xi = f \qquad (10)$$

$$\eta = \omega f = \sum_{n=1}^{\infty} \omega_n e^n f = e\omega_1 f + e^2\omega_2 f + \dots \qquad (11)$$

where the constants ω_n are as yet undetermined. Since x and y are now functions of the independent variables ξ and η,

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their derivatives become

$$d/df = \partial/\partial\xi + \omega\partial/\partial n \quad (12)$$

$$d^2/df^2 = \partial^2/d\xi^2 + 2\omega(\partial^2/\partial\xi\partial\eta) + \omega^2(\partial^2/\partial\eta^2) \quad (13)$$

Now x, y, and g are expanded in a power series of the eccentricity e;

$$x(f) = \sum_{n=0}^{\infty} x_n(f)e^n = x_0(f) + ex_1(f) + \ldots \quad (14a)$$

$$y(f) = \sum_{n=0}^{\infty} y_n(f)e^n = y_0(f) + ey_1(f) + \ldots \quad (14b)$$

$$g(e,f) = (1 + e\cos f)^{-1} = 1 - e\cos f + e^2\cos^2 f - \ldots \quad (15)$$

The mass ratio μ is also expanded in a power series about $\mu = \mu_0$ (where μ_0 indicates the value of μ under consideration, μ_a or μ_b);

$$\mu = \mu_0 + e\mu_1 + e^2\mu_2 + \ldots \quad (16)$$

Since f_1 and f_2 are functions of μ, their expansions become

$$f_1 = \sum_{n=0}^{\infty} a_n e^n = a_0 + ea_1 + e^2a_2 + \ldots$$
$$f_2 = \sum_{n=0}^{\infty} b_n e^n = b_0 + eb_1 + e^2b_2 + \ldots \quad (17)$$

where

$$a_0 = \tfrac{3}{2}\{1 + [1 - 3\mu_0(1 - \mu_0)]^{1/2}\}$$
$$b_0 = \tfrac{3}{2}\{1 - [1 - 3\mu_0(1 - \mu_0)]^{1/2}\} \quad (18)$$

$$b_1 = -a_1 = (9/4\kappa)(1 - 2\mu_0)\mu_1 \quad (19)$$

$$b_2 = -a_2 = (9/4\kappa)\big((1 - 2\mu_0)\mu_2 - \{1 - \tfrac{3}{4}[(1 - 2\mu_0)/\kappa]^2\}\mu_1^2\big) \quad (20)$$

$$\kappa = [1 - 3\mu_0(1 - \mu_0)]^{1/2} \quad (21)$$

Substituting Eqs. (10–17) into the equations of motion (8) and equating to zero the coefficients of like powers of e gives the following equations:

1st order (e^0)

$$x_{0\xi\xi} - 2y_{0\xi} - b_0x_0 = 0 \quad (22a)$$

$$y_{0\xi\xi} + 2x_{0\xi} - a_0y_0 = 0 \quad (22b)$$

2nd order (e^1)

$$x_{1\xi\xi} - 2y_{1\xi} - b_0x_1 = -2\omega_1x_{0\xi\eta} + 2\omega_1y_{0\eta} + b_1x_0 - b_0x_0\cos\xi \quad (23a)$$

$$y_{1\xi\xi} + 2x_{1\xi} - a_0y_1 = -2\omega_1y_{0\xi\eta} - 2\omega_1x_{0\eta} + a_1y_0 - a_0y_0\cos\xi \quad (23b)$$

3rd order (e^2)

$$x_{2\xi\xi} - 2y_{2\xi} - b_0x_2 = -2\omega_1x_{1\xi\eta} - 2\omega_2x_{0\xi\eta} - \omega_1^2x_{0\eta\eta} + 2\omega_1y_{1\eta} + b_2x_0 + 2\omega_2y_{0\eta} + b_1x_1 - b_0x_1\cos\xi - b_1x_0\cos\xi + b_0x_0\cos^2\xi \quad (24a)$$

$$y_{2\xi\xi} + 2x_{2\xi} - a_0y_2 = -2\omega_1y_{1\xi\eta} - 2\omega_2y_{0\xi\eta} - \omega_1^2y_{0\eta\eta} - 2\omega_1x_{1\eta} + a_2y_0 - 2\omega_2x_{0\eta} + a_1y_1 - a_0y_1\cos\xi - a_1y_0\cos\xi + a_0y_0\cos^2\xi \quad (24b)$$

where the subscripts of ξ and η denote partial differentiation.

4. Stability Analysis at $\mu = \mu_b$

Setting $\mu_0 = \mu_b = \frac{1}{2}[1 - (\frac{24}{27})^{1/2}] = 0.02859\ldots$, the solution of the first-order equations (22) becomes

$$x_0 = A_{01}(\eta)\cos\lambda_1\xi + B_{01}(\eta)\sin\lambda_1\xi + A_{02}(\eta)\cos\lambda_2\xi + B_{02}(\eta)\sin\lambda_2\xi \quad (25a)$$

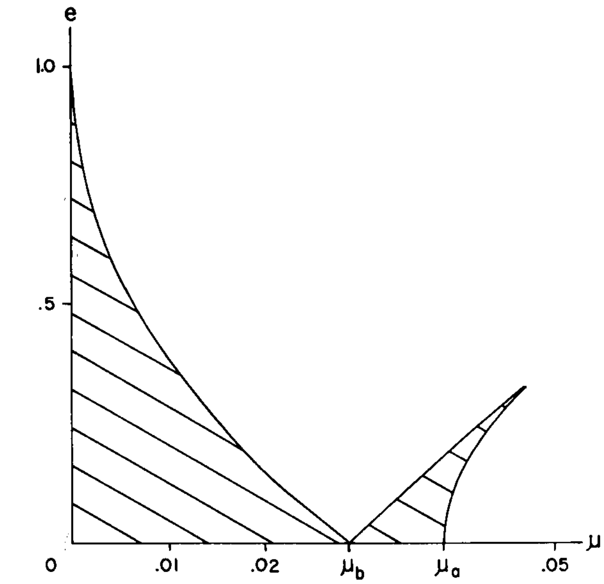

Fig. 1 Transition curves for infinitesimal motions about the triangular libration points in the elliptic restricted problem of three bodies.[1]

$$y_0 = \alpha_1B_{01}(\eta)\cos\lambda_1\xi - \alpha_1A_{01}(\eta)\sin\lambda_1\xi + \alpha_2B_{02}(\eta)\cos\lambda_2\xi - \alpha_2A_{02}\sin\lambda_2\xi \quad (25b)$$

where

$$\alpha_i = 2\lambda_i/(\lambda_i^2 + a_0) = (\lambda_i^2 + b_0)/2\lambda_i \quad (26)$$

and the λ_i are the roots of

$$(\lambda_i^2 + a_0)(\lambda_i^2 + b_0) - 4\lambda_i^2 = 0 \quad (27)$$

i.e., $\lambda_1 = \frac{1}{2}$, $\lambda_2 = (3)^{1/2}/2$. This solution is of the same form as the solution for the circular problem (Ref. 5, p. 257), except here the A_{0i} and B_{0i} are functions of η, since Eqs. (22) are partial differential equations. (In the circular problem, the A_{0i} and B_{0i} are constants.)

Substituting the first-order solution, Eqs. (25), into the second-order equations (23), and using $1 - \lambda_1 = \lambda_1$, yields

$$x_{1\xi\xi} - 2y_{1\xi} - b_0x_1 = C_1\cos\lambda_1\xi + D_1\sin\lambda_1\xi + C_2\cos\lambda_2\xi + D_2\sin\lambda_2\xi - (b_0/2)A_{01}\cos(1 + \lambda_1)\xi - (b_0/2)B_{01}\sin(1 + \lambda_1)\xi - (b_0/2)A_{02}\cos(1 + \lambda_2)\xi - (b_0/2)B_{02}\sin(1 + \lambda_2)\xi - (b_0/2)A_{02}\cos(1 - \lambda_2)\xi + (b_0/2)B_{02}\sin(1 - \lambda_2)\xi \quad (28a)$$

$$y_{1\xi\xi} + 2x_{1\xi} - a_0y_1 = E_1\cos\lambda_1\xi + F_1\sin\lambda_1\xi + E_2\cos\lambda_2\xi + F_2\sin\lambda_2\xi - (a_0/2)\alpha_1B_{01}\cos(1 + \lambda_1)\xi + (a_0/2)\alpha_1A_{01}\sin(1 + \lambda_1)\xi - (a_0/2)\alpha_2B_{02}\cos(1 + \lambda_2)\xi + (a_0/2)\alpha_2A_{02}\sin(1 + \lambda_2)\xi - (a_0/2)\alpha_2B_{02}\cos(1 - \lambda_2)\xi - (a_0/2)\alpha_2A_{02}\sin(1 - \lambda_2)\xi \quad (28b)$$

where

$$C_1 = 2\omega_1(\alpha_1 - \lambda_1)B_{01\eta} + (b_1 - b_0/2)A_{01}$$
$$D_1 = -2\omega_1(\alpha_1 - \lambda_1)A_{01\eta} + (b_1 + b_0/2)B_{01}$$
$$C_2 = 2\omega_1(\alpha_2 - \lambda_2)B_{02\eta} + b_1A_{02}$$
$$D_2 = -2\omega_1(\alpha_2 - \lambda_2)A_{02\eta} + b_1B_{02} \quad (29)$$
$$E_1 = 2\omega_1(\alpha_1\lambda_1 - 1)A_{01\eta} + \alpha_1(a_1 - a_0/2)B_{01}$$
$$F_1 = 2\omega_1(\alpha_1\lambda_1 - 1)B_{01\eta} - \alpha_1(a_1 + a_0/2)A_{01}$$
$$E_2 = 2\omega_1(\alpha_2\lambda_2 - 1)A_{02\eta} + \alpha_2a_1B_{02}$$
$$F_2 = 2\omega_1(\alpha_2\lambda_2 - 1)B_{02\eta} - \alpha_2a_1A_{02}$$

The $\cos\lambda_1\xi$, $\sin\lambda_1\xi$, $\cos\lambda_2\xi$, $\sin\lambda_2\xi$ terms in Eqs. (28) produce secular terms in x_1 and y_1. For a uniformly valid solution, x_1 and y_1 must be bounded for all times, therefore the secular terms must be eliminated. The conditions which must be satisfied for there to be no secular terms in x_1 and y_1 are[4]

$$D_i = -\alpha_i E_i, \qquad i = 1, 2 \tag{30}$$

$$C_i = \alpha_i F_i, \qquad i = 1, 2 \tag{31}$$

Consider the elimination of the λ_1 secular terms first. Substitution of (29) into (30) and (31) for $i = 1$ gives

$$\begin{aligned} &2\omega_1(2\alpha_1 - \lambda_1 - \alpha_1^2\lambda_1)B_{01\eta} + \\ &\qquad [b_1 - b_0/2 + \alpha_1^2(a_1 + a_0/2)]A_{01} = 0 \\ &-2\omega_1(2\alpha_1 - \lambda_1 - \alpha_1^2\lambda_1)A_{01\eta} + \\ &\qquad [b_1 + b_0/2 + \alpha_1^2(a_1 - a_0/2)]B_{01} = 0 \end{aligned} \tag{32}$$

Assuming solutions of the form $e^{s\eta}$, the characteristic equation of the system (32) is

$$s^2 + \{[b_1^2(1 - \alpha_1^2)^2 - \tfrac{1}{4}(\alpha_1^2 a_0 - b_0)^2] \div [2\omega_1(2\alpha_1 - \lambda_1 - \alpha_1^2\lambda_1)]^2\} = 0 \tag{33}$$

where $a_1 = -b_1$ has been used. For $A_{01}(\eta)$ and $B_{01}(\eta)$ to be oscillatory, the roots of this equation must be imaginary. Therefore

$$b_1^2 > \tfrac{1}{4}(b_0 - \alpha_1^2 a_0)^2/(1 - \alpha_1^2)^2 \tag{34}$$

Substituting for a_0, b_0 and b_1 gives

$$\pm\ \mu_1 > \mu^* \tag{35}$$

where

$$\mu^* = [\kappa/3(1 - 2\mu_b)]\{\kappa[(1 + \alpha_1^2)/(1 - \alpha_1^2)] - 1\} = \tfrac{1}{24}(\tfrac{11}{6})^{1/2} \tag{36}$$

and

$$\kappa = [1 - 3\mu_b(1 - \mu_b)]^{1/2} \tag{37}$$

It must also be shown that the λ_2 secular terms can be eliminated. Substitution of (29) into (30) and (31) for $i = 2$ yields

$$\begin{aligned} 2\omega_1(2\alpha_2 - \lambda_2 - \alpha_2^2\lambda_2)B_{02\eta} + (b_1 + \alpha_2^2 a_1)A_{02} = 0 \\ -2\omega_1(2\alpha_2 - \lambda_2 - \alpha_2^2\lambda_2)A_{02\eta} + (b_1 + \alpha_2^2 a_1)B_{02} = 0 \end{aligned} \tag{38}$$

Assuming solutions of the form $e^{p\eta}$, the characteristic equation of the system (38) is

$$p^2 + [b_1(1 - \alpha_2^2)/2\omega_1(2\alpha_2 - \lambda_2 - \alpha_2^2\lambda_2)]^2 = 0 \tag{39}$$

The roots of this equation are imaginary; hence A_{02} and B_{02} are oscillatory terms and no instability occurs.

From Eqs. (33) and (39) it is obvious that the only requirement on ω_1 is that it must be unequal to zero. The numerical value of ω_1 is arbitrary; therefore it will be taken as unity. The solution of x_1 and y_1 is

$$x_1 = A_{11}(\eta)\cos\lambda_1\xi + B_{11}(\eta)\sin\lambda_1\xi + A_{12}(\eta)\cos\lambda_2\xi + B_{12}(\eta)\sin\lambda_2\xi + \sum_{j=1}^{3}(M_j\cos\sigma_j\xi + N_j\sin\sigma_j\xi) \tag{40a}$$

$$\begin{aligned} y_1 = {}& \alpha_1\left[B_{11}(\eta) - \frac{E_1}{2\lambda_1}\right]\cos\lambda_1\xi - \alpha_1\left[A_{11}(\eta) + \frac{F_1}{2\lambda_1}\right]\sin\lambda_1\xi + \\ & \alpha_2\left[B_{12}(\eta) - \frac{E_2}{2\lambda_2}\right]\cos\lambda_2\xi - \alpha_2\left[A_{12}(\eta) + \frac{F_2}{2\lambda_2}\right]\sin\lambda_2\xi + \\ & \sum_{j=1}^{3}[P_j\cos\sigma_j\xi + Q_j\sin\sigma_j\xi] \end{aligned} \tag{40b}$$

where

$$\sigma_1 = (1 + \lambda_1), \qquad \sigma_2 = (1 + \lambda_2), \qquad \sigma_3 = (1 - \lambda_2) \tag{41}$$

and

$$M_1 = \left[\frac{b_0}{2}(\sigma_1^2 + a_0) + a_0\alpha_1\sigma_1\right]\frac{A_{01}}{\beta_1} = \bar{M}_1 A_{01},\ N_1 = \bar{M}_1 B_{01}$$

$$M_2 = \left[\frac{b_0}{2}(\sigma_2^2 + a_0) + a_0\alpha_2\sigma_2\right]\frac{A_{02}}{\beta_2} = \bar{M}_2 A_{02},\ N_2 = \bar{M}_2 B_{02}$$

$$M_3 = \left[\frac{b_0}{2}(\sigma_3^2 + a_0) - a_0\alpha_2\sigma_3\right]\frac{A_{02}}{\beta_3} = \bar{M}_3 A_{03},\ N_3 = -\bar{M}_3 B_{02}$$

$$P_1 = \left[b_0\sigma_1 + (\sigma_1^2 + b_0)\frac{a_0\alpha_1}{2}\right]\frac{B_{01}}{\beta_1} = \bar{P}_1 B_{01} \tag{42}$$

$$Q_1 = -\bar{P}_1 A_{01}$$

$$P_2 = \left[b_0\sigma_2 + (\sigma_2^2 + b_0)\frac{a_0\alpha_2}{2}\right]\frac{B_{02}}{\beta_2} = \bar{P}_2 B_{02},\ Q_2 = -\bar{P}_2 A_{02}$$

$$P_3 = \left[-\sigma_3 b_0 + (\sigma_3^2 + b_0)\frac{a_0\alpha_2}{2}\right]\frac{B_{02}}{\beta_3} = \bar{P}_3 B_{02},\ Q_3 = \bar{P}_3 A_{02}$$

where

$$\beta_i = [\sigma_i^2 - \lambda_1^2][\sigma_i^2 - \lambda_2^2] \tag{43}$$

Substituting the first- and second-order solutions Eqs. (25) and (40) into the third-order equations (24) gives

$$x_{2\xi\xi} - 2y_{2\xi} - b_0 x_2 = \tilde{C}_1\cos\lambda_1\xi + \tilde{D}_1\sin\lambda_1\xi + \tilde{C}_2\cos\lambda_2\xi + \tilde{D}_2\sin\lambda_2\xi + \text{terms which produce nonsecular terms} \tag{44a}$$

$$y_{2\xi\xi} + 2x_{2\xi} - a_0 y_2 = \tilde{E}_1\cos\lambda_1\xi + \tilde{F}_1\sin\lambda_1\xi + \tilde{E}_2\cos\lambda_2\xi + \tilde{F}_2\sin\lambda_2\xi + \text{terms which produce nonsecular terms} \tag{44b}$$

where $\tilde{C}_i$, $\tilde{D}_i$, $\tilde{E}_i$ and $\tilde{F}_i$ are functions of A_{0i}, B_{0i}, A_{1i}, B_{1i} and their derivatives.

For no secular terms in x_2 and y_2

$$\tilde{D}_i = -\alpha_i\tilde{E}_i \qquad i = 1,2 \tag{45}$$

$$\tilde{C}_i = \alpha_i\tilde{F}_i \qquad i = 1,2 \tag{46}$$

Consider the removal of the λ_1 secular terms first. Applying (45) and (46) for $i = 1$ gives

$$2(2\alpha_1 - \lambda_1 - \alpha_1^2\lambda_1)B_{11\eta} + \{b_1 - (b_0/2) + \alpha_1^2[a_1 + (a_0/2)]\}A_{11} = K_1 A_{01} \tag{47a}$$

$$-2(2\alpha_1 - \lambda_1 - \alpha_1^2\lambda_1)A_{11\eta} + \{b_1 + (b_0/2) + \alpha_1^2[a_1 - (a_0/2)]\}B_{11} = K_2 B_{01} \tag{47b}$$

where K_1 and K_2 are functions of ω_2, a_0, b_0, b_1, and b_2.

For a uniformly valid solution, A_{11} and B_{11} must be bounded functions for all f. The homogeneous portion of (47) is identical to the differential equation for A_{01} and B_{01}, Eq. (32). Hence the complimentary part of the solution of (47) is oscillatory if Eq. (35) is satisfied. However, the frequency of A_{01} and B_{01} is the same as the frequency of the complimentary parts of A_{11} and B_{11}. Therefore the nonhomogeneous portion of (47) will lead to secular terms in A_{11} and B_{11}. The removal of these secular terms will give an equation for μ_2. For no secular terms in A_{11} or B_{11}

$$\{b_1 + (b_0/2) + \alpha_1^2[a_1 - (a_0/2)]\}K_1 + \{b_1 - (b_0/2) + \alpha_1^2[a_1 + (a_0/2)]\}K_2 = 0 \tag{48}$$

Evaluating (48) on the transition curve gives the following value for b_2 (i.e., set $\mu_1^2 = \mu^{*2}$):

$$b_2 = \frac{1}{(1 - \alpha_1^2)}\left\{\left[\frac{\alpha_1^3}{\lambda_1}a_0 - (1 + \alpha_1^2)\right]\left|\frac{b_1}{2}\right| - \frac{1}{2}(b_0 + a_0\alpha_1^2) + \frac{\alpha_1^3}{2\lambda_1}\left(b_1^2 + \frac{a_0^3}{4}\right) + \frac{a_0\alpha_1}{2}\bar{P}_1 + \frac{b_0}{2}\bar{M}_1\right\} \tag{49}$$

$$b_2 = 23\kappa/704 = 0.0312\ \ldots \tag{50}$$

The corresponding value of μ_2 is

$$\mu_2 = 49(2)^{1/2}/4608 = 0.0150\ldots \quad (51)$$

For $i = 2$, Eqs. (45) and (46) remove the λ_2 secular terms from x_2 and y_2. These equations provide two nonhomogeneous ordinary differential equations for A_{12} and B_{12}. A_{02} and B_{02} will cause resonance in A_{12} and B_{12} just as A_{01} and B_{01} caused resonance in A_{11} and B_{11}. Removal of these resonance terms gives a value for ω_2, which does not influence the transition curves.

The equations of the transition curves intersecting the μ axis at $\mu = \mu_b$ are

$$\mu = \tfrac{1}{2}[1 - (\tfrac{24}{27})^{1/2}] \pm 24(\tfrac{1}{6})^{1/2}e + [49(2)^{1/2}/4608]e^2 + 0(e^3) \quad (52)$$

or

$$\mu = 0.02859\ldots \pm (0.05641\ldots)e + (0.01503\ldots)e^2 + 0(e^3) \quad (53)$$

On the basis of his numerical work Danby hypothesized that if either of these transition curves were reflected in the μ axis, then the reflection would form a continuation of the other curve that would be completely smooth. This is verified to $0(e^3)$, since the equations of the curves are the same, if the sign of e is changed.

5. Stability Analysis at $\mu = \mu_a$

For $\mu = \mu_a$, the solution for the circular orbit ($e = 0$) is unstable (see Ref. 5, p. 264). In order to satisfy arbitrary initial conditions, x and y are expanded in a power series of e beginning with e^{-1} terms:

$$x(f) = \frac{1}{e}\sum_{n=0}^{\infty} x_n(f)e^n = \frac{1}{e}[x_0(f) + ex_1(f) + \ldots] \quad (54a)$$

$$y(f) = \frac{1}{e}\sum_{n=0}^{\infty} y_n(f)e^n = \frac{1}{e}[y_0(f) + ey_1(f) + \ldots] \quad (54b)$$

(see Ref. 2, p. 235 or Ref. 6, p. 99). If Eqs. (54) replace Eqs. (14) in the perturbation scheme described by Eqs. (10–17), then Eqs. (22–24) emerge unchanged.

Setting $\mu_0 = \mu_a = \frac{1}{2}[1 - (\frac{23}{27})^{1/2}] = 0.03852\ldots$ the solution of the first-order equations (22) is

$$x_0 = A_{01}(\eta)\cos\lambda\xi + B_{01}(\eta)\sin\lambda\xi + A_{02}(\eta)\xi\cos\lambda\xi + B_{02}(\eta)\xi\sin\lambda\xi \quad (55a)$$

$$y_0 = \bar{A}_{01}(\eta)\cos\lambda\xi + \bar{B}_{01}(\eta)\sin\lambda\xi + \bar{A}_{02}(\eta)\xi\cos\lambda\xi + \bar{B}_{02}(\eta)\xi\sin\lambda\xi \quad (55b)$$

where

$$\bar{A}_{02} = \alpha B_{02}, \qquad \bar{B}_{02} = -\alpha A_{02} \quad (56)$$
$$\bar{A}_{01} = \alpha(B_{01} + A_{02}), \qquad \bar{B}_{01} = -\alpha(A_{01} - B_{02})$$

and

$$\lambda = 1/(2)^{1/2}, \qquad \alpha = (\lambda^2 + b_0)/2\lambda = 2\lambda/(\lambda^2 + a_0) \quad (57)$$

For a stable solution it is necessary to choose

$$A_{02}(\eta) \equiv B_{02}(\eta) \equiv 0 \quad (58)$$

Substitution of the first-order solution, Eqs. (55), into the second-order equations (23) gives

$$x_{1\xi\xi} - 2y_{1\xi} - b_0x_1 = G_1\cos\lambda\xi + H_1\sin\lambda\xi - (b_0/2)\{A_{01}[\cos(1+\lambda)\xi + \cos(1-\lambda)\xi] + B_{01}[\sin(1+\lambda)\xi - \sin(1-\lambda)\xi]\} \quad (59a)$$

$$y_{1\xi\xi} + 2x_{1\xi} - a_0y_1 = G_2\cos\lambda\xi + H_2\sin\lambda\xi - (a_0/2)\{\bar{A}_{01}[\cos(1+\lambda)\xi + \cos(1-\lambda)\xi] + \bar{B}_{01}[\sin(1+\lambda)\xi - \sin(1-\lambda)\xi]\} \quad (59b)$$

where

$$G_1 = 2\omega_1[\bar{A}_{01\eta} - \lambda B_{01\eta}] + b_1A_{01}$$
$$H_1 = 2\omega_1[\bar{B}_{01\eta} + \lambda A_{01\eta}] + b_1B_{01} \quad (60)$$
$$G_2 = -2\omega_1[A_{01\eta} + \lambda\bar{B}_{01\eta}] + a_1\bar{A}_{01}$$
$$H_2 = -2\omega_1[B_{01\eta} - \lambda\bar{A}_{01\eta}] + a_1\bar{B}_{01}$$

x_1 and y_1 will contain secular terms of the form $\xi\cos\lambda\xi$, $\xi\sin\lambda\xi$, $\xi^2\cos\lambda\xi$, $\xi^2\sin\lambda\xi$. For a uniformly valid solution these secular terms must be eliminated. For no $\xi^2\cos\lambda\xi$ and $\xi^2\sin\lambda\xi$ terms

$$H_1 + \alpha G_2 = 0, \qquad G_1 - \alpha H_2 = 0 \quad (61)$$

or

$$b_1(1 - \alpha^2)A_{01} = 0, \qquad -b_1(1 - \alpha^2)B_{01} = 0 \quad (62)$$

Therefore b_1 must vanish and hence

$$\mu_1 = 0 \quad (63)$$

The solutions of (59) then become

$$x_1 = A_{11}(\eta)\cos\lambda\xi + B_{11}(\eta)\sin\lambda\xi + A_{12}(\eta)\xi\cos\lambda\xi + B_{12}(\eta)\xi\sin\lambda\xi + \sum_{j=1}^{2}(T_{1j}\cos\sigma_j\xi + U_{1j}\sin\sigma_j\xi) \quad (64a)$$

$$y_1 = \bar{A}_{11}(\eta)\cos\lambda\xi + \bar{B}_{11}(\eta)\sin\lambda\xi + \bar{A}_{12}(\eta)\xi\cos\lambda\xi + \bar{B}_{12}(\eta)\xi\sin\lambda\xi + \sum_{j=1}^{2}(T_{2j}\cos\sigma_j\xi + U_{2j}\sin\sigma_j\xi) \quad (64b)$$

where

$$\sigma_1 = \lambda + 1, \qquad \sigma_2 = \lambda - 1$$
$$\bar{A}_{12} = \alpha B_{12}, \qquad \bar{B}_{12} = -\alpha A_{12} \quad (65)$$
$$\bar{A}_{11} = \alpha(B_{11} + A_{12}), \quad \bar{B}_{11} = -\alpha(A_{11} - B_{12})$$

and

$$T_{1j} = R_{1j}A_{01}(\eta), \qquad T_{2j} = R_{2j}B_{01}(\eta) \quad (66)$$
$$U_{1j} = R_{1j}B_{01}(\eta), \qquad U_{2j} = -R_{2j}A_{01}(\eta)$$

where

$$R_{1j} = [(\sigma_j^2 + a_0)(b_0/2) + a_0\alpha\sigma_j]/\beta_j$$
$$R_{2j} = [b_0\sigma_j + (a_0/2)\alpha(\sigma_j^2 + b_0)]/\beta_j \quad (67)$$
$$\beta_j = (\sigma_j^2 - \lambda^2)^2$$

For a stable solution, it is necessary to choose

$$A_{12}(\eta) \equiv B_{12}(\eta) \equiv 0 \quad (68)$$

Substitution of the second-order solution, Eqs. (64), and the first-order solution, Eqs. (55), into the third-order equations (24) gives

$$x_{2\xi\xi} - 2y_{2\xi} - b_0x_2 = \tilde{G}_1\cos\lambda\xi + \tilde{H}_1\sin\lambda\xi + \text{terms which produce a uniformly valid } x_2 \text{ and } y_2 \quad (69a)$$

$$y_{2\xi\xi} + 2x_{2\xi} - a_0y_2 = \tilde{G}_2\cos\lambda\xi + \tilde{H}_2\sin\lambda\xi + \text{terms which produce a uniformly valid } x_2 \text{ and } y_2 \quad (69b)$$

where $\tilde{G}_i$ and $\tilde{H}_i$ are known functions of A_{01}, B_{01}, A_{11}, and B_{11} and their derivatives.

x_2 and y_2 will contain secular terms of the form $\xi\cos\lambda\xi$, $\xi\sin\lambda\xi$, $\xi^2\cos\lambda\xi$, $\xi^2\sin\lambda\xi$. For a uniformly valid solution these secular terms must be eliminated. For no $\xi^2\cos\lambda\xi$

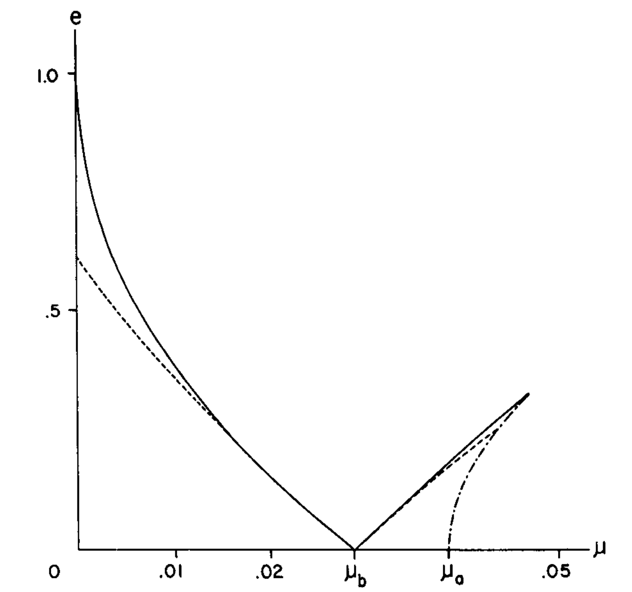

Fig. 2 Comparison of results of this paper with the numerical results obtained by Danby.[1] Solid line represents Danby's results. Dashed line represents Eq. (80). Dot-dashed line represents Danby's results and Eq. (81), which are indistinguishable.

and $\xi^2 \sin\lambda\xi$ terms

$$\tilde{H}_1 + \alpha \tilde{G}_2 = 0, \qquad \tilde{G}_1 - \alpha\tilde{H}_2 = 0 \tag{70}$$

or

$$B_{01\eta\eta} + \gamma^2(b_2{}^* - b_2)B_{01} = 0, \quad A_{01\eta\eta} + \gamma^2(b_2{}^* - b_2)A_{01} = 0 \tag{71}$$

where

$$2(1 - \alpha^2)b_2{}^* = -(b_0 + \alpha^2 a_0) + b_0(R_{11} + R_{12}) + \alpha a_0(R_{21} + R_{22}) \qquad \text{i.e., } b_2{}^* = (2)^{1/2}/8 \tag{72}$$

and where $\gamma = \text{const.}$

Hence for stable solutions

$$b_2 < b_2{}^* \tag{73}$$

The value of μ_2 corresponding to the transition curve ($b_2 = b_2{}^*$) becomes

$$\mu_2 = 2/(23)^{1/2}(27)^{1/2} = 0.08025\ldots \tag{74}$$

Thus the equation of the transition curve intersecting the $\mu = \mu_a$ is

$$\mu = \tfrac{1}{2}[1 - (23/27)^{1/2}] + [2/(23)^{1/2}(27)^{1/2}]e^2 + 0(e^3) \tag{75}$$

$$\mu = 0.03852\ldots + (0.08025\ldots)e^2 + 0(e^3) \tag{76}$$

6. Summary

The stability of infinitesimal motions about the triangular libration points in the elliptic restricted problem of three bodies has been investigated using the two variable expansion method.[2,3,6] The results of this investigation are the following analytical expressions for the transition curves bounding regions of stability:

$$\mu = \tfrac{1}{2}[1 - (\tfrac{24}{27})^{1/2}] \pm 24(\tfrac{11}{6})^{1/2}e + [49(2)^{1/2}/4608]e^2 + 0(e^3) \tag{80}$$

$$\mu = \tfrac{1}{2}[1 - (\tfrac{23}{27})^{1/2}] + [2/(23)^{1/2}(27)^{1/2}]e^2 + 0(e^3) \tag{81}$$

These curves are shown in Fig. 2, where they are compared with Danby's numerical results. Concerning the curves which intersect the μ axis at $\mu = \mu_b$, Danby hypothesized that if either of these curves were reflected in the μ axis, then the reflection would form a continuation of the other curve that would be completely smooth. This is verified to $0(e^3)$ since the equations of the curves are the same if the sign of e is changed.

References

[1] Danby, J. M. A., "Stability of the Triangular Points in the Elliptic Restricted Problem of Three Bodies," *Astronomical Journal*, Vol. 69, No. 2, 1964, pp. 165–172.

[2] Kevorkian, J., "The Two Variable Expansion Procedure for the Approximate Solution of Certain Nonlinear Differential Equations," *Space Mathematics*, Vol. 3, 1966, pp. 206–275.

[3] Cochran, J. A., "Problems in Singular Perturbation Theory," TR 1, 1962, Applied Mathematics and Statistics Lab., Stanford Univ.

[4] Alfriend, K. T. and Rand, R. H., "A Perturbation Solution for the Stability of the Triangular Points in the Elliptic Restricted Problem," *Journal of the Astronautical Sciences*, May-June 1968, pp. 105–109.

[5] Szebehely, V., *Theory of Orbits*, Academic Press, New York, 1967.

[6] Cole, J. D., *Perturbation Methods in Applied Mathematics*, Blaisdell, Waltham, Mass., 1968.