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A pair of van der Pol oscillators coupled by fractional derivatives

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Abstract We consider the stability of the in-phase and out-of-phase modes of a pair of fractionally-coupled van der Pol oscillators:

$$x'' - \epsilon(1 - x^2)x' + x = \epsilon \gamma D^\alpha(y - x) \quad (1)$$

$$y'' - \epsilon(1 - y^2)y' + y = \epsilon \gamma D^\alpha(x - y) \quad (2)$$

where $D^\alpha x$ is the order α derivative of $x(t)$, and $0 < \alpha < 1$. We use a two-variable perturbation method on the system's corresponding variational equations to derive expressions for the transition curves separating regions of stability from instability in the α, γ parameter plane. The perturbation results are validated with numerics and through direct comparison with known results in the limiting cases of $\alpha = 0$ and $\alpha = 1$, where the fractional coupling reduces to position coupling and velocity coupling, respectively.

Keywords Fractional calculus · Fractional derivative · Coupled oscillators · Perturbation methods

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1 Introduction

An increasingly important topic in the literature of engineering, science, and applied mathematics is that of fractional calculus and fractional differential equations. It is attractive to use fractional calculus in modeling phenomena that depend both on the current state and the overall time history. Application areas include viscoelasticity, electromagnetics, heat conduction, control theory, and diffusion [1–7]. Recent literature has dealt with the treatment of diverse fractional differential equations. These include fractional linear oscillators [2, 8–12], a fractional Duffing equation [8, 13], fractional van der Pol type equations [14–16], a fractional Mathieu equation [17], a fractional jerk model [18], a fractional wave equation [2], and equations exhibiting chaos [19–21].

It is the purpose of the present work to extend the treatment of a pair of coupled van der Pol oscillators. The coupling scheme used in previous treatments of coupled van der Pols involved both springlike coupling and damping coupling. This may be thought of as being due to coupling by a rheological material which exhibits both elasticity and dissipation. Such a system might be more realistically modeled by combining the effects of stiffness and damping into a single term consisting of a fractional derivative. Previous works [22, 23, 29] have considered the stability of the in-phase and out-of-phase modes of a system of van der Pol oscillator with varying types of coupling. Consider the system investigated by [22], where two van der Pol oscillators are connected through both position-coupling and velocity-coupling terms:

$$x'' - \epsilon(1-x^2)x' + x = \epsilon A(y-x) + \epsilon B(y'-x') \tag{3}$$

$$y'' - \epsilon(1-y^2)y' + y = \epsilon A(x-y) + \epsilon B(x'-y') \tag{4}$$

We instead consider the case of fractional-coupling,

$$x'' - \epsilon(1-x^2)x' + x = \epsilon\gamma D^\alpha(y-x) \tag{5}$$

$$y'' - \epsilon(1-y^2)y' + y = \epsilon\gamma D^\alpha(x-y) \tag{6}$$

In the limits as $\alpha \to 0$ and $\alpha \to 1$, the fractional-coupling respectively reduces to position coupling and velocity coupling. We look for the convergence of our results with those of previous works in these limiting cases of α.

We begin the paper with a brief introduction to the fractional calculus. See, e.g., [1, 24–27].

2 Fractional Derivatives

We begin with an intuitive definition for the fractional derivative of t^k, $D^\alpha t^k$. From there we derive an integral expression for the fractional derivative. Issues of convergence are ignored in our derivation, and it may therefore be thought of as a plausibility argument instead of a rigorous derivation. As shown in Ross [25], we note that

$$\frac{d^m}{dt^m}t^n = \frac{n!}{(n-m)!}t^{n-m} \tag{7}$$

where $m \le n$ are positive integers. Note that (7) can be written in terms of the gamma function $\Gamma(n+1) = n!$:

$$\frac{d^m}{dt^m}t^n = \frac{\Gamma(n+1)}{\Gamma(n-m+1)}t^{n-m} \tag{8}$$

By using the gamma function we can now generalize (8) to include all positive real numbers by replacing n by k and m by α, where k and α are positive real numbers, and we obtain:

$$D^\alpha t^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}t^{k-\alpha} \tag{9}$$

As an example, we compute the order $1/2$ derivative of t:

$$D^{1/2}t = \frac{\Gamma(2)}{\Gamma(3/2)}t^{1/2} = \frac{2}{\sqrt{\pi}}t^{1/2} \tag{10}$$

By the law of exponents of derivatives,

$$D^{1/2}D^{1/2}t = D^{1/2+1/2}t = \frac{d}{dt}t = 1 \tag{11}$$

Using this result from the law of exponents, we check (10) by taking the order $1/2$ derivative of it:

$$D^{1/2}\frac{2}{\sqrt{\pi}}t^{1/2} = \frac{2}{\sqrt{\pi}}D^{1/2}t^{1/2} = \frac{2}{\sqrt{\pi}}\frac{\Gamma(3/2)}{\Gamma(1)}t^0 = 1 \tag{12}$$

Now suppose that we have a function $x(t)$ which is expandable in a Taylor series about $t = 0$,

$$x(t) = \sum \frac{x^{(k)}(0)}{k!}t^k \tag{13}$$

where $x^{(k)}(0)$ is the kth derivative of x evaluated at $t = 0$. Taking the fractional derivative of both sides, we have

$$\begin{aligned} D^\alpha x(t) &= \sum \frac{x^{(k)}(0)}{k!}D^\alpha t^k \\ &= \sum \frac{x^{(k)}(0)}{k!}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}t^{k-\alpha} \end{aligned} \tag{14}$$

Following the treatment in Ross [25], we note that

$$\begin{aligned} \int_0^t (t-u)^m u^n\,du &= \frac{m!\,n!}{(m+n+1)!}t^{m+n+1} \\ &= \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}t^{m+n+1} \end{aligned} \tag{15}$$

We look to use (15) in simplifying (14) and hence $t^{k-\alpha} = t^{m+n+1}$. This yields an appropriate change of variables $n = k$ and $m = -1-\alpha$, and (15) becomes

$$\int_0^t (t-u)^{-1-\alpha}u^k\,du = \frac{\Gamma(-\alpha)\,\Gamma(k+1)}{\Gamma(k-\alpha+1)}t^{k-\alpha} \tag{16}$$

Solving for the common term appearing in (14), we have

$$\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}t^{k-\alpha} = \frac{1}{\Gamma(-\alpha)}\int_0^t (t-u)^{-1-\alpha}u^k\,du \tag{17}$$

Substituting (17) into (14), we obtain

$$D^\alpha x(t) = \sum \frac{x^{(k)}(0)}{k!}\frac{1}{\Gamma(-\alpha)}\int_0^t (t-u)^{-1-\alpha}u^k\,du \tag{18}$$

Interchanging the processes of summation and integration, we obtain

$$D^\alpha x(t) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-u)^{-1-\alpha} \left\{ \sum \frac{x^{(k)}(0) u^k}{k!} \right\} du \tag{19}$$

Recognizing our original Taylor expansion for $x(t)$, (13), we obtain

$$D^\alpha x(t) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-u)^{-1-\alpha} x(u)\, du \tag{20}$$

To avoid divergence of the Gamma function in (20), we use a trick from Ross [25]. From the law of exponents of derivatives we write

$$D^\alpha x(t) = D^m D^{-p} x(t) \tag{21}$$

where $\alpha = m - p$, $0 < p < 1$, and m is the least integer larger than α. Using (20), we obtain

$$D^\alpha x(t) = \frac{d^m}{dt^m} \frac{1}{\Gamma(p)} \int_0^t (t-u)^{p-1} x(u)\, du \tag{22}$$

For the case of $0 < \alpha < 1$, we have that $m = 1$ and $p = 1 - \alpha$, giving

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-u)^{-\alpha} x(u)\, du \tag{23}$$

For example,

$$D^{1/2} x(t) = \frac{1}{\Gamma(1/2)} \frac{d}{dt} \int_0^t (t-u)^{-1/2} x(u)\, du \tag{24}$$

As a check, we use this formula to compute the order $1/2$ derivative of t:

$$\begin{aligned} D^{1/2} t &= \frac{1}{\Gamma(1/2)} \frac{d}{dt} \int_0^t (t-u)^{-1/2} u\, du \\ &= \frac{1}{\Gamma(1/2)} \frac{d}{dt} \left(\frac{4}{3} t^{3/2} \right) = \frac{2}{\sqrt{\pi}} t^{1/2} \end{aligned} \tag{25}$$

which agrees with (10). Equation (23) can be simplified by taking $v = t - u$, giving

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t v^{-\alpha} x(t-v)\, dv \tag{26}$$

Carrying out the differentiation under the integral sign, we obtain

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t v^{-\alpha} x'(t-v)\, dv + \frac{x(0)}{t^\alpha} \right) \tag{27}$$

Following the treatment in Ross [25, p. 17], we adopt the convention that $x(0) = 0$, giving the final formula which is the Riemann–Liouville definition for the fractional derivative:

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t v^{-\alpha} x'(t-v)\, dv \tag{28}$$

3 Stability of the in-phase mode

The system under consideration is composed of two van der Pol oscillators coupled by the fractional derivatives of their positions,

$$x'' - \epsilon(1 - x^2) x' + x = \epsilon \gamma D^\alpha (y - x) \tag{29}$$

$$y'' - \epsilon(1 - y^2) y' + y = \epsilon \gamma D^\alpha (x - y) \tag{30}$$

There exists an in-phase manifold defined by $x = y$. On this manifold the coupling term vanishes, and the system is reduced to two identical van der Pol oscillators. An approximate solution for this mode exists and is the limit cycle of the uncoupled van der Pol equation. To determine the stability of the in-phase mode, we look at small disturbances from it. The variational equations govern the evolution of these small disturbances. To obtain the variational equations, we introduce the small deviations $\phi = x - u$ and $\psi = y - u$ where u is the in-phase mode $u(t) = x(t) = y(t)$. Substituting these expressions into (29)–(30) and ignoring nonlinear terms yields

$$\begin{aligned} &\phi'' - \epsilon(1 - u^2) \phi' + (1 + 2\epsilon u u')\phi \\ &\quad = \epsilon \gamma D^\alpha (\psi - \phi) \end{aligned} \tag{31}$$

$$\begin{aligned} &\psi'' - \epsilon(1 - u^2) \psi' + (1 + 2\epsilon u u')\psi \\ &\quad = \epsilon \gamma D^\alpha (\phi - \psi) \end{aligned} \tag{32}$$

Defining the quantities $w = \phi + \psi$ and $v = \phi - \psi$ uncouples these equations:

$$w'' - \epsilon(1 - u^2) w' + (1 + \epsilon 2 u u') w = 0 \tag{33}$$

$$v'' - \epsilon(1 - u^2) v' + (1 + \epsilon 2 u u') v = -2\epsilon \gamma D^\alpha v \tag{34}$$

Observe that (33) is also the variational equation corresponding to a single van der Pol oscillator, meaning that it is obtained when one considers a small deviation from the limit cycle of the van der Pol equation. This then implies that within our system of fractionally-coupled van der Pols, (33) governs deviations from the in-phase mode that themselves lie on the in-phase manifold. Since the limit cycle of the van der Pol equation is known to be orbitally stable, we follow [22] and conclude that (33) does not cause instability of the in-phase solution within our coupled system.

Equation (34) governs deviations transverse to the in-phase manifold. We examine whether this type of deviation will cause instability by employing a two-variable perturbation method. We begin the perturbation method by defining

$$\xi = \omega t, \qquad \eta = \epsilon t \tag{35}$$

where ω is the power series expansion in ϵ for the frequency of the van der Pol's limit cycle as given by (16) in Storti [22]:

$$\omega = 1 + O(\epsilon^2) \tag{36}$$

By the chain rule the quantities v' and v'' become

$$v' = \omega v_\xi + \epsilon v_\eta \tag{37}$$

$$v'' = \omega^2 v_{\xi\xi} + 2\omega\epsilon v_{\xi\eta} + \epsilon^2 v_{\eta\eta} \tag{38}$$

Recalling (28) for $0 < \alpha < 1$, we define the fractional derivative as

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t z^{-\alpha} x'(t-z)\, dz \tag{39}$$

Osler [28] has shown that, for a function of the form $x(\xi)$ where $\xi = \omega t$,

$$D_t^\alpha x(\xi) = \omega^\alpha D_\xi^\alpha x(\xi) \tag{40}$$

We assume that, for functions of two arguments $x(\xi, \eta)$ where $\eta = \epsilon t$,

$$D_t^\alpha x(\xi,\eta)$$

$$= \omega^\alpha D_\xi^\alpha x(\xi, \eta) + o(1) \tag{41}$$

$$= \frac{\omega^\alpha}{\Gamma(1-\alpha)} \int_0^\xi z^{-\alpha} x_\xi(\xi - z, \eta)\, dz + o(1) \tag{42}$$

From (36)–(38) and (42), in terms of ξ and η, (34) becomes

$$\begin{aligned} &v_{\xi\xi} + 2\epsilon v_{\xi\eta} - \epsilon(1-u^2) v_\xi + (1 + 2\epsilon u u_\xi) v \\ &\quad + O(\epsilon^2) \\ &\quad = -2\epsilon\gamma \frac{1}{\Gamma(1-\alpha)} \int_0^\xi z^{-\alpha} x_\xi(\xi - z, \eta) dz + o(\epsilon) \end{aligned} \tag{43}$$

The in-phase mode, the van der Pol limit cycle, can be approximated by power series in ϵ:

$$u = 2\cos\xi + O(\epsilon) \tag{44}$$

We posit a power series solution for v:

$$v(\xi,\eta) = v_0(\xi,\eta) + \epsilon v_1(\xi,\eta) + \cdots \tag{45}$$

Substituting these two power series expressions (44)–(45) into (43) and collecting terms, we obtain

$$O(1): \quad \mathcal{L} v_0 = 0 \tag{46}$$

$$\begin{aligned} O(\epsilon): \quad \mathcal{L} v_1 &= -2 v_{0_{\xi\eta}} - (1 + 2\cos\xi) v_{0_\xi} \\ &\quad + (4\sin 2\xi) v - 2\gamma D_\xi^\alpha v_0 \end{aligned} \tag{47}$$

where $\mathcal{L}$ is the linear operator, $\mathcal{L} v_0 = v_{0_{\xi\xi}} + v_0$. The general solution to (46) is given by

$$v_0 = A(\eta)\cos\xi + B(\eta)\sin\xi \tag{48}$$

This general solution is substituted into the $O(\epsilon)$ equation (47), and resonant terms are identified. Denoting the nonresonant terms as NRT, we have

$$\begin{aligned} \mathcal{L} v_1 &= 2A\sin\xi + 2A'\sin\xi - 2B'\cos\xi \\ &\quad - 2\gamma D_\xi^\alpha \big(A(\eta)\cos\xi + B(\eta)\sin\xi\big) + \text{NRT} \end{aligned} \tag{49}$$

The fractional derivative (39) cannot be computed in closed form. Instead we approximate the fractional derivative by evaluating the integral in the limit as $\xi \to \infty$ and therefore expect our results to be valid for steady state [8, 14]:

$$\begin{aligned} D_\xi^\alpha v_0(\xi,\eta) &= \frac{1}{\Gamma(1-\alpha)} \int_0^\xi z^{-\alpha} \big(-A(\eta)\sin(\xi - z) \\ &\quad + B(\eta)\cos(\xi - z)\big)\, dz \qquad (50) \\ &= \frac{\cos\xi}{\Gamma(1-\alpha)} \int_0^\xi z^{-\alpha} \big(A(\eta)\sin z \end{aligned}$$

$$+ B(\eta)\cos z\big)\,dz$$
$$+ \frac{\sin\xi}{\Gamma(1-\alpha)}\int_0^{\xi} z^{-\alpha}\big(-A(\eta)\cos z$$
$$+ B(\eta)\sin z\big)\,dz \tag{51}$$
$$= \frac{\cos\xi}{\Gamma(1-\alpha)}\big(A(\eta)I_s + B(\eta)I_c\big)$$
$$+ \frac{\sin\xi}{\Gamma(1-\alpha)}\big(-A(\eta)I_c + B(\eta)I_s\big) \tag{52}$$

where

$$I_c = \int_0^{\xi} z^{-\alpha}\cos z\,dz, \qquad I_s = \int_0^{\xi} z^{-\alpha}\sin z\,dz \tag{53}$$

In the limit as $t \to \infty$ these integrals become

$$\int_0^{\infty} z^{-\alpha}\cos z\,dz, = \Gamma(1-\alpha)\sin\frac{\alpha\pi}{2},$$
$$\int_0^{\infty} z^{-\alpha}\sin z\,dz = \Gamma(1-\alpha)\cos\frac{\alpha\pi}{2} \tag{54}$$

Combining the results of (52)–(54) yields the final expression for our approximation to $D_{\xi}^{\alpha} v_0(\xi,\eta)$,

$$D_{\xi}^{\alpha} v_0(\xi,\eta)$$
$$= \cos\xi\left(A(\eta)\cos\frac{\alpha\pi}{2} + B(\eta)\sin\frac{\alpha\pi}{2}\right)$$
$$+ \sin\xi\left(-A(\eta)\sin\frac{\alpha\pi}{2} + B(\eta)\cos\frac{\alpha\pi}{2}\right) \tag{55}$$

Applying (55) to (49) yields

$$\mathcal{L}v_1 = \left(2A + 2A' + 2\gamma\left(A\sin\frac{\alpha\pi}{2}\right.\right.$$
$$\left.\left. - B\cos\frac{\alpha\pi}{2}\right)\right)\sin\xi$$
$$+ \left(-2B' - 2\gamma\left(A\cos\frac{\alpha\pi}{2}\right.\right.$$
$$\left.\left. + B\sin\frac{\alpha\pi}{2}\right)\right)\cos\xi + \text{NRT} \tag{56}$$

We obtain the slow flow equations by setting the secular terms to zero:

$$A' = \left(-1 - \gamma\sin\frac{\alpha\pi}{2}\right)A + \gamma B\cos\frac{\alpha\pi}{2} \tag{57}$$
$$B' = -\gamma A\cos\frac{\alpha\pi}{2} - \gamma B\sin\frac{\alpha\pi}{2} \tag{58}$$

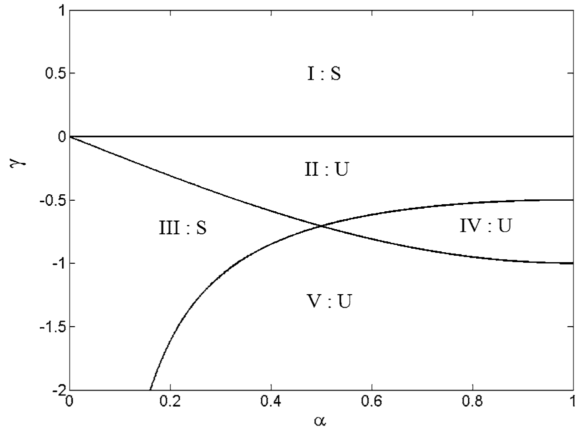

Fig. 1 Stability of the in-phase mode as predicted by the perturbation method. S denotes stable and U unstable. Regions I and III are both stable and composed of nodes and foci. Regions II, IV, and V are unstable. Regions II and IV are filled with saddles, and region V is composed of unstable nodes and foci, cf. (61)–(62)

This is a linear system of equations and can be written in matrix form,

$$\begin{bmatrix} A' \\ B' \end{bmatrix} = [M]\begin{bmatrix} A \\ B \end{bmatrix} \tag{59}$$

with coefficient matrix

$$M = \begin{bmatrix} -1 - \gamma\sin\frac{\alpha\pi}{2} & \gamma\cos\frac{\alpha\pi}{2} \\ -\gamma\cos\frac{\alpha\pi}{2} & -\gamma\sin\frac{\alpha\pi}{2} \end{bmatrix} \tag{60}$$

We categorize the stability of the system by analyzing the trace, tr, and determinant, det, of the coefficient matrix M:

$$tr = -1 - 2\gamma\sin\frac{\alpha\pi}{2} \tag{61}$$
$$det = \gamma\left(\gamma + \sin\frac{\alpha\pi}{2}\right) \tag{62}$$

There are two ways in which the system can change stability: one when $det = 0$ with $tr < 0$ and, secondly, when $tr = 0$ with $det > 0$, which corresponds to a Hopf bifurcation. The critical transition curves for stability $tr = 0$ and $det = 0$ are plotted in Fig. 1, and the regions of stability and instability are found.

These results are checked by numerically integrating (34) with u given by (44) and $\epsilon = 0.1$. A large number of discrete points in the α vs. γ parameter space are chosen, and the system is then integrated for each γ, α pair and checked for fulfillment of our

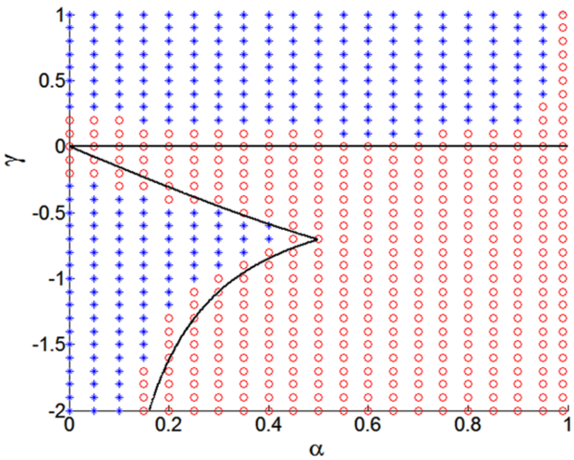

Fig. 2 Comparison of perturbation method's stability results for the in-phase mode previously shown in Fig. 1 with numerical integration of (34) with u given by (44) and ϵ taken as $\epsilon = .01$. The *asterisks* represent stable parameter values, and the *circles* unstable

chosen criterion for stability. Both the perturbation method and numerical results are shown in Fig. 2. The asterisks represent stable parameter values, and the circles unstable. These results are in good agreement with our perturbation method's results previously shown in Fig. 1 in that they both predict a stability change near the line $\gamma = 0$ as well as a wedge of stability in the $\gamma < 0$ region.

Our results also agree with those of Storti and Reinhall [22] in the limiting cases of $\alpha = 0$ and $\alpha = 1$, which respectively correspond to position coupling and velocity coupling. In the case of $\alpha = 0$, we find the in-phase mode to be stable for all values of the coupling coefficient. However note that "all values" mean all values for which the perturbation method is valid and therefore can reasonably be classified as coefficients of $O(1)$. This agrees with [22], where they find a loss of stability at a coupling coefficient of $-0.5/\epsilon$ which is beyond the scope of our perturbation method. For the case of $\alpha = 1$, our results agree with [22] in that stability of the in-phase mode is lost as the coupling coefficient transitions from a positive value to a negative one.

4 Stability of the out-of-phase mode

Next we consider the stability of the out-of-phase mode. The out-of-phase mode is characterized by the motions where $x = -y = q$. Substituting $x = -y = q$ into each of our coupled van der Pol equations, (29) and (30), they admit identical equations. This one equation, (63), must therefore be satisfied for the out-of-phase mode to exist:

$$q'' - \epsilon\left(1 - q^2\right)q' + q = -2\epsilon\gamma D^\alpha q \tag{63}$$

We seek to determine for which values of parameters γ and α will (63) exhibit periodic motions. A two-variable expansion is applied to answer this question. Periodic solutions can then be identified as fixed points in the slow flow equations produced from applying this perturbation method. Mirroring our in-phase analysis, we begin by again defining $\xi = \omega t$, $\eta = \epsilon t$, and $\omega = 1 + O(\epsilon^2)$ as in (35)–(36). By the chain rule, q' and q'' become

$$q' = \omega q_\xi + \epsilon q_\eta \tag{64}$$

$$q'' = \omega^2 q_{\xi\xi} + 2\omega\epsilon q_{\xi\eta} + \epsilon^2 q_{\eta\eta} \tag{65}$$

Recall that from (41) with $\omega = 1 + O(\epsilon^2)$ the fractional derivative is given by

$$D_t^\alpha q(\xi, \eta) = D_\xi^\alpha q(\xi, \eta) + o(1) \tag{66}$$

We posit a power series solution for q:

$$q(\xi, \eta) = q_0(\xi, \eta) + \epsilon q_1(\xi, \eta) + O\left(\epsilon^2\right) \tag{67}$$

Substituting (64)–(66) into the equation governing the out-of-phase motion (63) and collecting terms, we obtain:

$$O(1): \quad \mathcal{L}q_0 = 0 \tag{68}$$

$$O(\epsilon): \quad \mathcal{L}q_1 = -2q_{0_{\xi\eta}} + \left(1 - q_0^2\right)q_{0_\xi} - 2\gamma D_\xi^\alpha q_0 \tag{69}$$

where $\mathcal{L}$ is the linear operator, $\mathcal{L}q_0 = q_{0_{\xi\xi}} + q_0$. The general solution to (68) is given by

$$q_0 = A(\eta)\cos\xi + B(\eta)\sin\xi \tag{70}$$

Substituting this general solution into the $O(\epsilon)$ (69), we collect the resonant terms and identify nonresonant terms as NRT:

$$\mathcal{L}v_1 = \sin\xi\left(\frac{AB^2}{4} + 2A' + \frac{A^3}{4} - A\right) + \cos\xi\left(-2B' - \frac{B^3}{4} - \frac{A^2B}{4} + B\right)$$

$$- 2\gamma\, D_\xi^\alpha\big(A(\eta)\cos\xi + B(\eta)\sin\xi\big) + \text{NRT} \tag{71}$$

Recall our approximation for the fractional derivative (55),

$$D_\xi^\alpha q_0(\xi,\eta) = \cos\xi\left(A(\eta)\cos\frac{\alpha\pi}{2} + B(\eta)\sin\frac{\alpha\pi}{2}\right) + \sin\xi\left(-A(\eta)\sin\frac{\alpha\pi}{2} + B(\eta)\cos\frac{\alpha\pi}{2}\right) \tag{72}$$

Substituting this expression into (71), the resonant terms are collected and set to zero to obtain the slow flow equations:

$$A' = -\frac{A B^2}{8} - \frac{A^3}{8} + \frac{A}{2} - \gamma A\sin\frac{\alpha\pi}{2} + \gamma B\cos\frac{\alpha\pi}{2} \tag{73}$$

$$B' = -\frac{A^2 B}{8} - \frac{B^3}{8} + \frac{B}{2} - \gamma B\sin\frac{\alpha\pi}{2} - \gamma A\cos\frac{\alpha\pi}{2} \tag{74}$$

To uncouple the slow flow equations, we transform to polar coordinates:

$$q_0 = R(\eta)\cos\big(t - \theta(\eta)\big) \tag{75}$$

$$R' = -\frac{R^3 + R(8\gamma\sin\frac{\alpha\pi}{2} - 4)}{8} \tag{76}$$

$$\theta' = -\gamma\cos\frac{\alpha\pi}{2} \tag{77}$$

Since θ' equals a constant, it will not change the periodic nature of q_0 in (75), but it will alter the period. Periodic solutions are then seen to correspond to fixed points in the R slow flow equation. Solving $R' = 0$ for the amplitude R of these periodic solutions yields

$$R = 2\left(1 - 2\gamma\sin\frac{\alpha\pi}{2}\right)^{1/2} \tag{78}$$

This amplitude and hence the out-of-phase solution will exist for parameter γ and α pairs satisfying

$$1 - 2\gamma\sin\frac{\alpha\pi}{2} > 0 \tag{79}$$

This result is shown in Fig. 3 with the parameter plane divided into two regions, where in each region

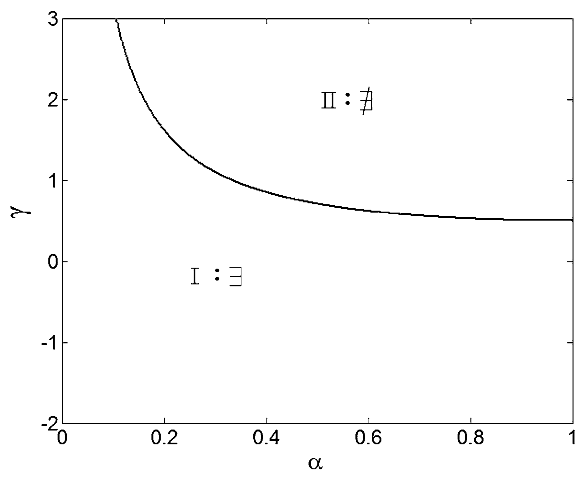

Fig. 3 The perturbation method predicts the out-of-phase mode to exist only in region I as denoted by ∃, cf. (79). The out-of-phase mode does not exist in region II as noted in the figure by ∄ symbol

the out-of-phase mode either exists or does not exist, and the transition curve is given by $\sin\frac{\alpha\pi}{2} = \frac{1}{2\gamma}$, cf. (79). This out-of-phase mode existence result agrees with the results of Storti and Reinhall [22] in the limit of $\alpha \to 1$ where the fractional coupling reduces to the velocity coupling. For this velocity-coupling case, it is known that an out-of-phase motion will exist for a coupling coefficient less than 0.5, which is the result we recover in our analysis. At this critical coupling coefficient, the out-of-phase limit cycle is created or destroyed in a Hopf bifurcation. The perturbation analysis indicates this as we see that the limit amplitude grows from an initial amplitude of zero. It is also known that the out-of-phase motion will only exist in the position-coupling case for a coupling coefficient greater than $-0.5/\epsilon$. In this case the out-of-phase motion loses stability in an infinite period bifurcation. This bifurcation is not detected by the perturbation method since a coupling coefficient of $O(1/\epsilon)$ is beyond its region of validity.

Next we seek to determine the stability of the out-of-phase mode in the parameter region of existence. The stability will again be studied through the corresponding variational equations, and we begin by introducing the small deviations $\phi = x - q$ and $\psi = -y - q$ where $q(t)$ is the out-of-phase mode. Substituting these expressions into the coupled van der Pol equations (29)–(30) and ignoring nonlinear terms yields

$$\phi'' - \epsilon\left(1-q^2\right)\phi' + (1+2\epsilon\, q\, q')\phi = -\epsilon\,\gamma\, D^{\alpha}(\psi+\phi) \tag{80}$$

$$\psi'' - \epsilon\left(1-q^2\right)\psi' + (1+2\epsilon\, q\, q')\psi = -\epsilon\,\gamma\, D^{\alpha}(\phi+\psi) \tag{81}$$

Defining the quantities $w=\phi+\psi$ and $v=\phi-\psi$ uncouples the equations:

$$w'' - \epsilon\left(1-q^2\right)w' + \left(1+\epsilon\, 2\, q\, q'\right)w = -2\,\epsilon\,\gamma\, D^{\alpha} w \tag{82}$$

$$v'' - \epsilon\left(1-q^2\right)v' + \left(1+\epsilon\, 2\, q\, q'\right)v = 0 \tag{83}$$

Observe that (82) is also the variational equation corresponding to (63), which is the equation that the out-of-phase mode satisfies. The equation on w therefore represents small deviations in the out-of-phase plane $x=-y$. We can determine if these w type deviations will give rise to instability from the previous two-variable analysis by considering the stability of the fixed points of (76). Let δR be a small deviation from R_* where R_* is the fixed point of (76). Substituting $R=R_*+\delta R$ into (76) and Tayloring the expression with respect to δR about zero yields

$$\delta R' = -\frac{3\,R_*^2 + 8\,\gamma\,\sin\frac{\alpha\pi}{2} - 4}{8}\,\delta R + O\left(\delta R^2\right) \tag{84}$$

The fixed point R_* satisfies (78), which reduces (84) to the form

$$\delta R' = -\frac{R_*^2}{4}\,\delta R + O\left(\delta R^2\right) \tag{85}$$

The fixed points of the slow flow are therefore asymptotically stable, and deviations that lie on the out-of-phase plane will not cause the out-of-phase mode to become unstable.

Returning now to the v variational equation, (83), we perform a two-variable expansion to determine if deviations transverse to the plane of the out-of-phase mode will cause instability. Again we begin by defining $\xi=\omega t$, $\eta=\epsilon\, t$, and $\omega=1+O(\epsilon^2)$ as in (35)–(36). By the chain rule, v' and v'' become

$$v' = \omega\, v_{\xi} + \epsilon\, v_{\eta} \tag{86}$$

$$v'' = \omega^2\, v_{\xi\,\xi} + 2\,\omega\,\epsilon\, v_{\xi\,\eta} + \epsilon^2\, v_{\eta\,\eta} \tag{87}$$

We posit a power series solution for v,

$$v(\xi,\ \eta) = v_0(\xi,\ \eta) + \epsilon\, v_1(\xi,\ \eta) + O\left(\epsilon^2\right) \tag{88}$$

The out-of-phase mode q is approximated as the limit cycle found in the previous two-variable perturbation method on (63):

$$q = R(\eta)\cos\left(t-\psi(\eta)\right) \tag{89}$$

$$= \left(\left(4 - 8\,\gamma\,\sin\frac{\alpha\,\pi}{2}\right)^{1/2} + O(\epsilon)\right) \times \cos\left(t - \epsilon\,\gamma\,\cos\frac{\alpha\pi}{2}t + O\left(\epsilon^2\right)\right) \tag{90}$$

$$= \left(4 - 8\,\gamma\,\sin\frac{\alpha\,\pi}{2}\right)^{1/2}\cos t + O(\epsilon) \tag{91}$$

$$q^2 = R^2(\eta)\cos^2\left(t-\psi(\eta)\right) \tag{92}$$

$$= \left(4 - 8\,\gamma\,\sin\frac{\alpha\,\pi}{2} + O(\epsilon)\right) \times \cos^2\left(t - \epsilon\,\gamma\,\cos\frac{\alpha\pi}{2}t + O\left(\epsilon^2\right)\right) \tag{93}$$

$$= \left(2 - 4\,\gamma\,\sin\frac{\alpha\,\pi}{2} + O(\epsilon)\right) \times \left(1+\cos\left(2t+O(\epsilon)\right)\right) \tag{94}$$

$$= \left(2 - 4\,\gamma\,\sin\frac{\alpha\,\pi}{2}\right)(1+\cos 2t) + O(\epsilon) \tag{95}$$

Substituting (86)–(95) into the v variational equation (83) and collecting terms, we obtain:

$$O(1):\quad \mathcal{L}v_0 = 0 \tag{96}$$

$$O(\epsilon):\quad \mathcal{L}v_1 = \left(1 + 8\gamma\,\sin\frac{\alpha\pi}{2}\cos^2 t - 4\cos^2 t\right)v_{0_\xi} - 2\,v_{0_{\eta\xi}} + \left(4 - 8\gamma\,\sin\frac{\alpha\pi}{2}\right) \times \cos t\ \sin t\ v_0 \tag{97}$$

where $\mathcal{L}$ is the linear operator, $\mathcal{L}v_0 = v_{0_{\xi\xi}} + v_0$. The general solution to (68) is given by

$$v_0 = A(\eta)\cos\xi + B(\eta)\sin\xi \tag{98}$$

Substituting this general solution into the $O(\epsilon)$ equation (69) and grouping and identifying nonresonant terms as NRT, we have

$$\mathcal{L}v_1 = \sin\xi\left(2A' - 4\gamma\sin\frac{\alpha\pi}{2}A + A\right) + \cos\xi\left(-2B' + 4\gamma\sin\frac{\alpha\pi}{2}B - B\right) + \text{NRT} \tag{99}$$

The resonant terms are set to zero to obtain the slow flow equations

$$A' = \frac{A}{2}\left(4\gamma\,\sin\frac{\alpha\pi}{2} - 1\right) \tag{100}$$

$$B' = \frac{B}{2}\left(4\gamma\,\sin\frac{\alpha\pi}{2} - 1\right) \tag{101}$$

From the slow flow equations we see that there is only one criterion for system stability,

$$\sin\frac{\alpha\pi}{2} < \frac{1}{4\gamma} \tag{102}$$

In Fig. 4 the upper curve divides the parameter plane into regions where the out-of-phase mode exists and regions where it does not exist as given by $\sin\frac{\alpha\pi}{2} = \frac{1}{2\gamma}$, cf. (79), and was previously shown in Fig. 3. The lower curve of Fig. 4 divides the region where the out-of-phase mode exists into stable and unstable regions and is given by $\sin\frac{\alpha\pi}{2} = \frac{1}{4\gamma}$, cf. (102).

The perturbation method's results are again compared with numerics. The results of repeated integration of (83) with q given by (91), $\epsilon = 0.1$, and for a large number of discrete α, γ parameter pairs are shown in Fig. 5. The asterisks represent stable parameter values, and the circles unstable. The numerics are in good agreement with our perturbation method's results.

Comparing our results again with those of Storti and Reinhall [22] in the limiting cases of $\alpha = 0$ and $\alpha = 1$, we find good agreement. For $\alpha = 0$, we find the out-of-phase mode to be stable for all values of the coupling coefficient in the region of validity of our perturbation method. This agrees with [22] as they find the out-of-phase mode to be stable for all parameter values greater than $0.5/\epsilon$, which is beyond the scope of our perturbation method. For the case of $\alpha = 1$, as the coupling coefficient is decreased from a value larger than 0.5 to a negative value, Storti and Reinhall [22] find that the out-of-phase mode transitions from not existing to being unstable and then finally stable. This transition trend is also seen in our results.

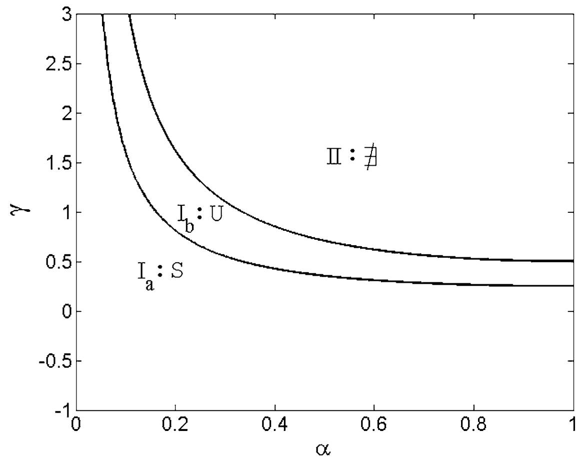

Fig. 4 The perturbation method predicts the out-of-phase motion to be stable in region Ia and unstable in region Ib, cf. (102). The out-of-phase motion does not exist in region II, cf. (79). S is stable, U is unstable, and ∄ means "does not exist"

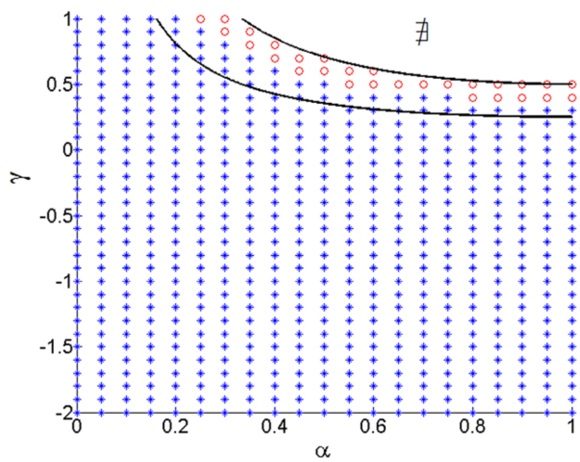

Fig. 5 Comparison of the out-of-phase perturbation method's results with numerical integration of (83) with q given by (91) and $\epsilon = 0.1$. The *asterisks* represent stable parameter values, and the *circles* unstable. ∄ represents that the out-of-phase mode does not exist

5 Conclusion

In this paper we considered a system of two van der Pol oscillators connected through fractional coupling. There exists an in-phase mode and an out-of-phase mode respectively defined by $x = y$ and $x = -y$. To determine the stability of a given mode, we looked at small disturbances from it. The variational equations govern the evolution of these small disturbances. We performed a two-variable perturbation method on the variational equations to determine if these small dis-

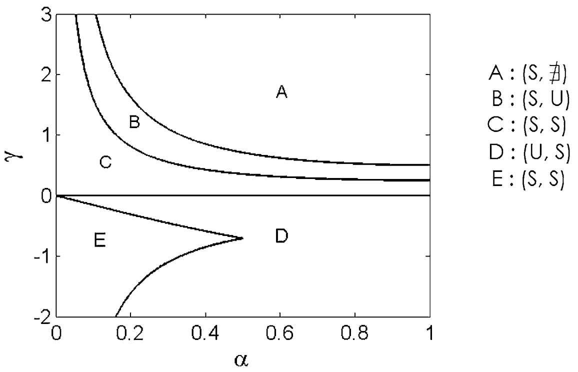

Fig. 6 Stability results for both the in-phase and out-of-phase modes. In region A, the in-phase mode is stable, and the out-of phase mode does not exist as denoted by $(S,\ \nexists)$. In region B, the in-phase mode is stable, and the out-of-phase mode is unstable, $(S,\ U)$. In region C, both the in-phase and out-of-phase modes are stable, $(S,\ S)$. In region D, the in-phase mode is unstable, and the out-of-phase mode is stable, $(U,\ S)$. In region E, both modes are again stable, $(S,\ S)$

turbances would grow or decay implying then that the mode is respectively unstable or stable.

Each mode had a corresponding set of variational equations composed of one equation governing disturbances in the plane of the mode and a second equation governing disturbances transverse to the plane of the mode. For both the in-phase and out-of-phase modes, instability was seen to only be caused by disturbances transverse to the plane.

The stability results for both modes are plotted in the α, γ parameter plane, Fig. 6. These transition curves were obtain from the two-variable perturbation method and were shown to be in good agreement with numerical integration. Additionally we have reconciled our results with those of [22] in the limiting cases of $\alpha = 0$ and $\alpha = 1$, where the fractional coupling reduces to position coupling and velocity coupling, respectively. Our work predicts that for all values of $0 \le \alpha \le 1$, there will be at least one region of bistability in the parameter plane meaning that both the in-phase and out-of-phase modes will be stable and the asymptotic behavior would then depend on initial conditions.

Appendix

The simple Euler's method is utilized in our numerical integration program, and to use this algorithm, we begin by expressing (34) as a system of first-order equations:

$$v_1' = v_2 \tag{103}$$

$$v_2' = \epsilon\left(1 - u^2\right)v_2 - \left(1 + 2\epsilon\, u\, u'\right)v_1 - \epsilon\, \gamma\, D^\alpha v_1 \tag{104}$$

Uniformly discretizing the interval $t = n\,h$, $n = 0, 1, 2, \ldots$, Euler's method gives

$$v_{1_{n+1}} = v_{1_n} + h\, v_{2_n} \tag{105}$$

$$v_{2_{n+1}} = v_{2_n} + h\Big(\epsilon\left(1 - u_n^2\right)v_{2_n} - \left(1 + 2\epsilon\, u_n\, u_n'\right)v_{1_n} - 2\epsilon\, \gamma\, D_n^\alpha v_1\Big) \tag{106}$$

The subscript n denotes that the variable is being evaluated at $t = n\,h$. What remains to be defined so that (105) and (106) can be implemented is the fractional derivative $D_n^\alpha v_1$. From (39) we have

$$D_n^\alpha v_1 = \frac{1}{\Gamma(1-\alpha)} \int_0^{n\,h} z^{-\alpha}\, v_1'(n\,h - z)\, dz \tag{107}$$

This integral expression is computed using the trapezoid rule. Note that the integrand becomes infinite at the lower integration bound and the trapezoid rule will fail. To avoid this divergence, we rewrite (108) using the additivity of definite integrals:

$$D_n^\alpha v_1 = \frac{1}{\Gamma(1-\alpha)} \left(\int_0^{h} z^{-\alpha}\, v_1'(n\,h - z)\, dz + \int_h^{n\,h} z^{-\alpha}\, v_1'(n\,h - z)\, dz \right) \tag{108}$$

We then approximate $z^{-\alpha}$ by $h^{-\alpha}$ in the first integral and evaluate

$$D_n^\alpha v_1 = \frac{1}{\Gamma(1-\alpha)} \left(h^{-\alpha}\left(v_{1_n} - v_{1_{n-1}}\right) + \int_h^{n\,h} z^{-\alpha}\, v_1'(n\,h - z)\, dz \right) \tag{109}$$

The second integral may now be computed using the trapezoid rule with z taking the same uniformly distributed values of t used in applying Euler's method:

$$D_n^\alpha v_1 = \frac{1}{\Gamma(1-\alpha)} \Big(h^{-\alpha}\left(v_{1_n} - v_{1_{n-1}}\right)$$

$$+\frac{h}{2}\Bigg(h^{-\alpha}v'_{1_{n-1}}+(n\,h)^{-\alpha}v'_{1_0}+\sum_{m=2}^{n-1}2\,(m\,h)^{-\alpha}\,v'_{1_{n-m}}\Bigg)\Bigg) \tag{110}$$

$$=\frac{1}{\Gamma(1-\alpha)}\Bigg(h^{-\alpha}(v_{1_n}-v_{1_{n-1}})+\frac{h}{2}\Bigg(h^{-\alpha}v_{2_{n-1}}+(n\,h)^{-\alpha}v_{2_0}+\sum_{m=2}^{n-1}2\,(m\,h)^{-\alpha}\,v_{2_{n-m}}\Bigg)\Bigg) \tag{111}$$

This summation will have to be calculated at each time step to then be used in (106). The sum is also growing at each step, and to ease the computation time, it can be written as a vector inner product, where the two vectors will be updated at each step:

$$h^{-\alpha}v_{2_{n-1}}+\sum_{m=2}^{n-1}2\,(m\,h)^{-\alpha}\,v_{2_{n-m}}+(n\,h)^{-\alpha}v_{2_0}=\bar{m}_{1_n}\bullet\bar{m}_{2_n} \tag{112}$$

$$\bar{m}_{1_n}=\left[h^{-\alpha},\ 2(2h)^{-\alpha},\ \ldots,\ 2((n-1)h)^{-\alpha},\ (nh)^{-\alpha}\right] \tag{113}$$

$$\bar{m}_{2_n}=\left[v_{2_{n-1}},\ v_{2_{n-2}},\ \ldots,\ v_{2_0}\right] \tag{114}$$

$$\bar{m}_{1_{n+1}}=\left[\bar{m}_{1_n}(1:n-1),\ 2(nh)^{-\alpha},\ \left((n+1)h\right)^{-\alpha}\right] \tag{115}$$

$$\bar{m}_{2_{n+1}}=\left[v_{2_n},\ \bar{m}_{2_n}\right] \tag{116}$$

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