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Perturbation Methods and Computer Algebra

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1 Introduction

The purpose of this paper is to report on some recent work aimed at automating the algebraic computations associated with the approximate solution of systems of nonlinear differential equations by perturbation methods. These methods, which have traditionally been performed by hand, are accomplished more accurately and more efficiently by using computer algebra. The programs which accomplish the following computations have been written in Macsyma [5] and will not be given here, but are available in [6]. We have implemented the following methods on Macsyma (see [6]):

- Linstedt's method
- Center manifold reductions
- Normal forms
- Two variable expansion method (multiple scales)
- Averaging
- Lie transforms for Hamiltonian systems
- Liapunov-Schmidt reductions

In this paper we shall apply the first three methods in the list to examples involving van der Pol's equation:

$$\frac{d^2 x}{dt^2} + x - e(1 - x^2) \frac{dx}{dt} = 0 \quad (1)$$

2 Lindstedt's Method

Van der Pol's equation possesses a periodic solution called a *limit cycle* for all $e \neq 0$ [2]. Lindstedt's method offers a scheme for approximating the limit cycle for small e . If in (1) we set $\tau = \omega t$, expand

$$x(\tau) = x_0(\tau) + ex_1(\tau) + \cdots + e^n x_n(\tau) \quad (2)$$

$$\omega = \omega_0 + e\omega_1 + \cdots + e^n \omega_n \quad (3)$$

and collect terms, we obtain a sequence of linear ODE's on the x_n 's, e.g.,

$$x_0'' + x_0 = 0 \quad (4)$$

$$x_1'' + x_1 = -2\omega_1 x_0'' + x_0'(1 - x_0^2). \quad (5)$$

Solving these recursively, we obtain at the n th step,

$$x_n'' + x_n = (\cdots) \sin \tau + (\cdots) \cos \tau + \text{non-resonant terms.} \quad (6)$$

Requiring the coefficients of $\sin \tau$ and $\cos \tau$ to vanish (in order to have a periodic solution) yields values for the amplitude and frequency of the limit cycle to order n . Here are typical results of the Macsyma program ([5],[6]). First we present (2) valid to $O(e^7)$:

$$x(\tau) = 2 \cos(\tau) - \frac{(\sin(3 \tau) - 3 \sin(\tau)) e}{4} \\ - \frac{(5 \cos(5 \tau) - 18 \cos(3 \tau) + 12 \cos(\tau)) e^2}{\cdots}$$

$$\begin{aligned}
& + (28 \sin(7 \tau) - 140 \sin(5 \tau) + 189 \sin(3 \tau) \\
& - 63 \sin(\tau)) e^3 / 2304 + (1647 \cos(9 \tau) \\
& - 10745 \cos(7 \tau) + 21700 \cos(5 \tau) \\
& - 16920 \cos(3 \tau) + 3285 \cos(\tau)) e^4 / 552960 \\
& - (49797 \sin(11 \tau) - 402678 \sin(9 \tau) \\
& + 1106210 \sin(7 \tau) - 1322125 \sin(5 \tau) \\
& + 582975 \sin(3 \tau) + 194565 \sin(\tau)) e^5 / 66355200 \\
& - (10728705 \cos(13 \tau) - 103688277 \cos(11 \tau) \\
& + 363275388 \cos(9 \tau) - 595527380 \cos(7 \tau) \\
& + 454903750 \cos(5 \tau) - 76288275 \cos(3 \tau) \\
& - 54423600 \cos(\tau)) e^6 / 55738368000 \\
& + (2340511875 \sin(15 \tau) - 26331793605 \sin(13 \tau) \\
& + 112564838232 \sin(11 \tau) - 237436059483 \sin(9 \tau) \\
& + 260002792805 \sin(7 \tau) - 118802031250 \sin(5 \tau) \\
& - 18287608500 \sin(3 \tau) + 34770385650 \sin(\tau)) e^7 \\
& / 46820229120000 + \dots
\end{aligned}$$

Next we display (3) valid to $O(e^8)$:

$$w = 1 - \frac{e^2}{16} + \frac{17 e^4}{3072} + \frac{35 e^6}{884736} - \frac{678899 e^8}{5096079360} + \dots$$

3 Center Manifold Reduction

Center manifold reduction is a technique which simplifies an appropriate problem by focusing attention on nonlinear extensions of critical eigendirections, thereby eliminating unimportant information from consideration. In order to provide an example of this method, we consider the behavior of the van der Pol equation at infinity, i.e., for large values of x and dx/dt [3]. Following Poincare (see [4]) we set in (1)

$$y = \frac{dx}{dt}, \quad v = \frac{y}{x}, \quad z = \frac{1}{x}, \quad d\tau = x^2 dt \quad (7)$$

to obtain

$$\begin{aligned} z' &= -vz^3 \\ v' &= -ev + z^2(ev - v^2 - 1). \end{aligned} \quad (8)$$

This system of DE's describes the behavior of van der Pol's equation at infinity, which here corresponds to $z = 0$ [3]. An equilibrium exists at the origin, $v = z = 0$. Linearization about the origin shows that the eigenvalues are $-e$ and 0 . Corresponding to the 0 eigenvalue is the eigendirection $z = 0$. The center manifold theorem [1] tells us that there exists a (generally curved) line (the center manifold) tangent to this eigendirection which is invariant under the flow (8). The flow along the center manifold determines the stability of the equilibrium point.

In order to obtain an approximate expression for the center manifold, we set

$$z = a_2 v^2 + a_3 v^3 + \dots \quad (9)$$

and substitute (9) into (8). Replacing derivatives of z and v by the values prescribed by the flow (8), and collecting terms, we obtain a sequence of algebraic equations on the a_i .

Here are typical results of the Macsyma program [6]. First we present the center manifold to $O(z^{12})$:

$$v = \frac{z^2}{e} - \frac{z^4}{e} + \frac{(e-1)z^2}{3} - \frac{(e-5)z^6}{3} + \dots$$

$$\begin{aligned}
 & \frac{(e^{-4} - 14e^{-2} + 6)z^{10}}{e^5} - \frac{(e^{-4} - 30e^{-2} + 54)z^{12}}{e^5} + \dots
 \end{aligned}$$

Next we display the flow on the center manifold to $O(z^{15})$:

$$\frac{dz}{dt} = \frac{z^5}{e} + \frac{z^7}{e} + \frac{(e^{-1} - 1)z^9}{3e} + \frac{(e^{-2} - 5)z^{11}}{3e}$$

$$\begin{aligned}
 & + \frac{(e^{-4} - 14e^{-2} + 6)z^{13}}{e^5} + \frac{(e^{-4} - 30e^{-2} + 54)z^{15}}{e^5} + \dots
 \end{aligned}$$

In particular, the last result gives that for $e > 0$ the equilibrium $v = z = 0$ is unstable.

4 Normal Forms

Normal forms is a method for obtaining approximate solutions to a system of ordinary differential equations. The method consists of generating a transformation of variables such that the transformed equations in the new variables are in a simplified form. As an example, we again consider the van der Pol equation at infinity. Since the transformation (7) is singular for $x = 0$, the foregoing analysis based on (7) and (8) is invalid near the “ends” of the y -axis. To complete our investigation of the van der Pol equation at infinity, we interchange the roles of x and y in (7). We set in (1)

$$y = \frac{dx}{dt}, \quad u = \frac{x}{y}, \quad z = \frac{1}{y}, \quad d\tau = y^2 dt \tag{10}$$

which gives

$$\begin{aligned}
 z' &= uz^3 + ez(u^2 - z^2) \\
 u' &= z^2(1 + u^2 - eu) + eu^3.
 \end{aligned} \tag{11}$$

As in (8), $z = 0$ here corresponds to the trajectory at infinity. An equilibrium exists in (11) at $u = z = 0$. We are interested in its stability. Although (11) have no linear part, the substitution

$$w = z^2 \quad (12)$$

produces the system:

$$\begin{aligned} w' &= 2uw^2 + 2ew(u^2 - w) \\ u' &= w(1 + u^2 - eu) + eu^3 \end{aligned} \quad (13)$$

which has the linear part

$$w' = 0, \quad u' = w. \quad (14)$$

Although the linearized system (14) has a double zero eigenvalue, and hence is not sufficient to yield conclusions about stability, the question can be settled by the method of normal forms. Takens [9] has shown that systems with this linear part can always be put in the normal form

$$\begin{aligned} p' &= a_2q^2 + a_3q^3 + a_4q^4 + \dots \\ q' &= p + b_2q^2 + b_3q^3 + b_4q^4 + \dots \end{aligned} \quad (15)$$

where p and q are related to w and u via a near-identity transformation.

Here is the result of the Macsyma program which computes the near-identity transformation and the new normal form (see [6],[8]): If the following near-identity transformation is substituted into (13),

$$\begin{aligned} [\bar{w} = p - 2e q p + \frac{(12e^2 + 3)q^2 p + 2e q^3}{3} \\ - \frac{(24e^3 + 15e^2)q^3 p + 7e^2 q^4}{3}, \end{aligned}$$

$$\begin{aligned}
 & + \frac{(960 e^4 + 1035 e^2 + 80) q^4 p + (370 e^3 + 48 e) q^5}{60} + \dots, \\
 u = q & - \frac{3 e^2 q^2}{2} + \frac{(15 e^2 + 4) q^3}{6} - \frac{(105 e^3 + 70 e) q^4}{24} \\
 & + \frac{(945 e^4 + 1086 e^2 + 88) q^5}{120} + \dots]
 \end{aligned}$$

the following transformed differential equations result:

$$\frac{dp}{dt} = -2 e^2 q^5 + \dots,$$

$$\frac{dq}{dt} = p + \frac{5 e^3 q^3}{3} - \frac{5 e^2 q^4}{2} + \frac{(225 e^3 + 8 e) q^5}{60} + \dots]$$

Have we taken enough terms in the series to correctly describe the local behavior of the system in the neighborhood of the origin? A truncated system is said to be *determined* if the inclusion of any higher order terms cannot effect the topological nature of the local behavior near the equilibrium. Using Macsyma, it has been shown [8] that a system of the form:

$$p' = a_5 q^5 + a_6 q^6 + a_7 q^7 + \dots \tag{16}$$

$$q' = p + b_3 q^3 + b_4 q^4 + b_5 q^5 + \dots$$

is topologically equivalent to the truncated system:

$$p' = a_5 q^5 \tag{17}$$

$$q' = p + b_3 q^3$$

if

$$a_5 \neq 0 \quad \text{and} \quad 4a_5 + 3b_3^2 > 0. \quad (18)$$

In the foregoing case of van der Pol's equation, (18) are satisfied, and hence the stability of the origin in (11) is the same as in the system:

$$\begin{aligned} p' &= -2e^2 q^5 \\ q' &= p + \frac{5}{3} e q^3. \end{aligned} \quad (19)$$

Equations (19) may be simplified by setting $r = q^3$ and reparameterizing time with $dT = q^2 d\tau$ (recall τ was defined in (10)), giving

$$\begin{aligned} p' &= -2e^2 r \\ r' &= 3p + 5er. \end{aligned} \quad (20)$$

Equations (20) are a linear system with eigenvalues $2e$ and $3e$, and thus the origin is unstable for $e > 0$ (and stable for $e < 0$).

A Appendix

We include in this Appendix the results of a normal form computation for the general two dimensional nilpotent case (of which (13) are a special case):

$$w' = \sum_i \sum_j g_{ij} u^i w^j, \quad u' = w + \sum_i \sum_j f_{ij} u^i w^j. \quad (21)$$

in which the summed terms are strictly nonlinear. Equations (21) can be transformed to Takens' normal form (15) by a near-identity transformation, which to $O(3)$ is given by:

$$[w = p + \frac{2 \quad 2}{2 \quad k1 \quad p \quad + \quad 2 \quad g02 \quad q \quad p \quad + \quad g11 \quad q}]{}$$

$$\begin{aligned}
 & + (6 k_3 p^3 + (6 g_{11} k_2 + 12 g_{02} k_1 + 6 g_{03}) q p^2 \\
 & + (6 g_{20} k_2 + 3 g_{11} f_{02} + 6 g_{11} k_1 + 6 g_{02}^2 + 3 g_{12}) \\
 & q^2 p + (4 g_{20} f_{02} + g_{11} f_{11} + (-2 f_{20} + 4 g_{11}) g_{02} \\
 & + 2 g_{21}) q^3)/6,
 \end{aligned}$$

$$u = q$$

$$\begin{aligned}
 & + \frac{2 k_2 p^2 + (2 f_{02} + 2 k_1) q p + (f_{11} + g_{02}) q^2}{2} \\
 & + (6 k_4 p^3 + (6 f_{11} k_2 + 12 k_1 f_{02} + 6 f_{03} + 6 k_3) q p^2 \\
 & + ((6 f_{20} + 3 g_{11}) k_2 + (3 f_{11} + 6 g_{02}) f_{02} \\
 & + (6 f_{11} + 6 g_{02}) k_1 + 3 f_{12} + 3 g_{03}) q^2 p \\
 & + (-2 g_{20} k_2 + (2 f_{20} + 2 g_{11}) f_{02} + (2 f_{20} + g_{11}) k_1 \\
 & + f_{11}^2 + 3 g_{02} f_{11} + 2 g_{02}^2 + 2 f_{21} + g_{12}) q^3)/6]
 \end{aligned}$$

The parameters k_1, k_2, k_3, k_4 in the above transformation are arbitrary constants resulting from a nonempty null space of the linear equations determining the coefficients of the near-identity transformation [6]. If the near-identity transformation is appropriately extended to terms of $O(5)$, the following normal form may be obtained:

$$\begin{aligned}
 \frac{dp}{dt} & = g_{20} q^2 + (f_{11} g_{20} - g_{11} f_{20} + g_{30}) q^3 \\
 & - (20 k_2 g_{20} + ((16 f_{02} - 8 k_1) f_{20}
 \end{aligned}$$

$$\begin{aligned}
& + (-2 f_{02} - 4 k_1) g_{11} - 7 f_{11}^2 - 6 g_{02} f_{11} + g_{02}^2 \\
& - 8 f_{21} + 2 g_{12}) g_{20} - 12 g_{02} f_{20} \\
& + ((6 f_{11} + 6 g_{02}) g_{11} + 12 g_{21}) f_{20} + 12 f_{30} g_{11} \\
& - 18 g_{30} f_{11} - 6 g_{02} g_{30} - 12 g_{40}) q / 12 \\
& - ((42 k_2 f_{11} - 10 k_2 g_{02} - 16 f_{02}^2 + 4 k_1 f_{02} - 4 k_1^2 \\
& + 16 f_{03} + 10 k_3) g_{20}^2 + ((-42 k_2 g_{11} + 12 f_{02} f_{11} \\
& + (28 f_{02} + 4 k_1) g_{02} + 12 f_{12} - 14 g_{03}) f_{20} \\
& + ((f_{02} - 2 k_1) f_{11} + (f_{02} - 2 k_1) g_{02} - 2 f_{12} \\
& - 4 g_{03}) g_{11} - 3 f_{11}^3 - 7 g_{02} f_{11}^2 \\
& + (-3 g_{02}^2 - 12 f_{21} + g_{12}) f_{11} + 42 k_2 g_{30} + g_{02}^3 \\
& + (-8 f_{21} + 3 g_{12}) g_{02} + (-2 f_{02} - 4 k_1) g_{21} \\
& + 12 f_{30} f_{02} - 12 f_{30} k_1 - 6 f_{31} + 2 g_{22}) g_{20} \\
& + (-12 f_{02} g_{11} - 12 g_{02}^2 - 12 g_{12}) f_{20}^2 \\
& + ((3 f_{11}^2 + 6 g_{02} f_{11} + 3 g_{02}^2) g_{11} + 12 g_{21} f_{11} \\
& + 24 f_{02} g_{30} + (12 g_{21} - 24 f_{30}) g_{02} + 12 g_{31}) f_{20} \\
& + (12 f_{30} f_{11} + 12 f_{30} g_{02} + 12 f_{40}) g_{11} - 15 g_{30} f_{11}^2 \\
& + (-18 g_{02} g_{30} - 24 g_{40}) f_{11} \\
& + (-3 g_{02}^2 - 12 f_{21}) g_{30} - 12 g_{40} g_{02} + 12 f_{30} g_{21}
\end{aligned}$$

$$\begin{aligned}
& - 12 g_{50}) q / 12 + \dots, \\
\frac{dq}{dt} &= p + \frac{(2 f_{20} + g_{11}) q^2}{2} \\
& - ((2 f_{02} + 6 k_1) g_{20} + 2 g_{02} f_{20} + (- f_{11} - g_{02}) g_{11} \\
& - 2 g_{21} - 6 f_{30}) q / 6 - ((16 k_2 f_{20} + 8 k_2 g_{11} \\
& + 18 k_1 f_{11} - 12 f_{02} g_{02} + 6 f_{12} + 18 g_{03}) g_{20} \\
& + (8 f_{02} + 8 k_1) f_{20}^2 + (- 10 k_1 g_{11} - 2 f_{11}^2 + 8 g_{02}^2 \\
& + 8 f_{21} + 10 g_{12}) f_{20} + (f_{02} + 2 k_1) g_{11}^2 \\
& + (- f_{11}^2 - 3 g_{02} f_{11} - 2 g_{02}^2 - 2 f_{21} - g_{12}) g_{11} \\
& + (- 6 g_{21} - 12 f_{30}) f_{11} + (6 f_{02} + 18 k_1) g_{30} \\
& + (- 6 g_{21} - 6 f_{30}) g_{02} - 6 g_{31} - 24 f_{40}) q / 24 \\
& + ((40 k_2 f_{02} + 120 k_2 k_1) g_{20}^2 \\
& + ((- 60 k_2 f_{11} - 100 k_2 g_{02} + 48 f_{02}^2 + 136 k_1 f_{02} \\
& - 104 k_1^2 + 8 f_{03} + 140 k_3) f_{20} \\
& + (- 50 k_2 f_{11} - 90 k_2 g_{02} + 4 f_{02}^2 + 8 k_1 f_{02} \\
& - 52 k_1^2 + 4 f_{03} + 70 k_3) g_{11} + (- 2 f_{02} - 42 k_1) f_{11}^2 \\
& + ((36 f_{02} - 36 k_1) g_{02} - 12 f_{12} - 72 g_{03}) f_{11} \\
& + (50 f_{02} + 6 k_1) g_{02}^2 - 72 g_{03} g_{02}
\end{aligned}$$

$$\begin{aligned}
& + (8 f_{02} - 48 k_1) f_{21} + (28 f_{02} + 12 k_1) g_{12} \\
& - 40 k_2 g_{21} - 120 k_2 f_{30} - 24 f_{22} - 36 g_{13}) g_{20} \\
& + (60 k_2 g_{11} + (- 32 f_{02} - 32 k_1) g_{02} + 44 g_{03}) f_{20}^2 \\
& + (30 k_2 g_{11}^2 + ((- 14 f_{02} + 16 k_1) f_{11} \\
& + (- 38 f_{02} + 36 k_1) g_{02} + 8 f_{12} + 50 g_{03}) g_{11} \\
& + 8 g_{02} f_{11}^2 - 22 g_{12} f_{11} - 60 k_2 g_{30} - 32 g_{02}^3 \\
& + (- 32 f_{21} - 74 g_{12}) g_{02} + (- 12 f_{02} + 32 k_1) g_{21} \\
& - 120 f_{30} f_{02} - 120 f_{30} k_1 - 60 f_{31} - 44 g_{22}) f_{20} \\
& + ((- 3 f_{02} - 10 k_1) f_{11} + (- 3 f_{02} - 10 k_1) g_{02} \\
& - 2 f_{12} - 4 g_{03}) g_{11}^2 + (f_{11}^3 + 6 g_{02} f_{11}^2 \\
& + (11 g_{02}^2 + 8 f_{21} + 5 g_{12}) f_{11} - 30 k_2 g_{30} + 6 g_{02}^3 \\
& + (12 f_{21} + 7 g_{12}) g_{02} + (- 10 f_{02} - 20 k_1) g_{21} \\
& - 12 f_{30} f_{02} + 12 f_{30} k_1 + 6 f_{31} + 2 g_{22}) g_{11} \\
& + (14 g_{21} + 30 f_{30}) f_{11}^2 + ((- 12 f_{02} - 108 k_1) g_{30} \\
& + (36 g_{21} + 48 f_{30}) g_{02} + 36 g_{31} + 120 f_{40}) f_{11} \\
& + ((36 f_{02} - 36 k_1) g_{02} - 24 f_{12} - 72 g_{03}) g_{30} \\
& + (22 g_{21} - 6 f_{30}) g_{02}^2 + (36 g_{31} + 96 f_{40}) g_{02} \\
& + 16 g_{21} f_{21} + (8 g_{21} - 24 f_{30}) g_{12} \\
& + (- 24 f_{02} - 72 k_1) g_{40} + 120 f_{50} + 24 g_{41}) q^5 / 120 \\
& + \dots]
\end{aligned}$$

Note that if $g_{20} \neq 0$, then k_1 may be chosen so as to kill the q^3 term in the q' equation. Similarly, if $(g_{11} + 2f_{20})g_{20} \neq 0$, then k_2 and k_3 may be respectively chosen to kill the q^4 and q^5 terms in the q' equation.

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