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BIFURCATION OF PERIODIC MOTIONS IN TWO WEAKLY COUPLED VAN DER POL OSCILLATORS

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Abstract We study a pair of weakly coupled van der Pol oscillators and investigate the bifurcations of phase-locked periodic motions which occur as the coupling coefficients are varied. Perturbation methods are used and their relation to the topological structure of solutions in the four dimensional phase space is discussed. While the problem is formulated for general linear coupling, the case of detuning plus diffusive coupling via displacement and velocity is discussed in more detail. It is shown that up to four phase-locked periodic motions can exist in this case.

1. INTRODUCTION

The dynamics of coupled nonlinear oscillators has received much attention over the last twenty years. In the case of *conservative* systems, R. M. Rosenberg and his associates have extensively investigated the properties of certain periodic motions called nonlinear normal modes. The engineering significance of these vibrations lies in the result that resonance in the forced system occurs in the neighborhood of the nonlinear normal modes of the unforced system [1, 2]. For a summary of results concerning the existence, approximation and stability of nonlinear normal modes the reader is referred to a well-known review article by Rosenberg [3] as well as to more recent works [4–6].

The present work is aimed at investigating the related topic of dynamics in a *nonconservative* system of coupled oscillators. In particular we shall study two identical van der Pol oscillators with weak linear coupling:

$$\ddot{\mathbf{u}} + \mathbf{u} - \varepsilon(\mathbf{I} - \mathbf{U}^2)\dot{\mathbf{u}} = \varepsilon(\mathbf{A}\mathbf{u} + \mathbf{B}\dot{\mathbf{u}}) \tag{1}$$

where

$$\mathbf{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{U} = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}, \quad \varepsilon \ll 1,$$

where **A**, **B** are 2×2 matrices with general coefficients, and where dots represent differentiation with respect to time t.

We will obtain approximate expressions for the motion of this system and will reduce the problem of finding phase-locked periodic motions (representing mutual entrainment) to the study of three algebraic equations.

In particular we will be interested in a special case of (1) which is motivated by biological applications. If x_1 and x_2 represent the states (e.g. the concentration of a chemical species) of two neighboring cells or groups of cells, each of which is able to oscillate by itself (see, e.g. Pavlidis [7, pp. 66–69]), and if solutes are permitted to flow between these oscillators by *diffusion*, then the coupling terms would be proportional to the differences $x_1 - x_2$ and $\dot{x}_1 - \dot{x}_2$. Such diffusive coupling corresponds to the following choices for matrices **A** and **B** of (1):

$$\mathbf{A} = \alpha\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\Delta \end{bmatrix}, \qquad \mathbf{B} = \beta\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{2}$$

Here α, β are diffusion coefficients and Δ is a parameter related to the difference in the frequencies of the uncoupled oscillators. (For small ε, the x_1 oscillator has frequency 1 while the x_2 oscillator has frequency $(1 + \varepsilon\Delta)^{1/2} \sim 1 + \varepsilon\Delta/2$.)

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For the special case of (2) we will present additional results concerning bifurcation and stability of periodic motions.

This work extends previous studies by other investigators, which we outline here. Minorsky [8] has considered the case of inertial coupling, essentially a special case of (2) above ($\beta = 0$); Hayashi and Kuramitsu [9] and Tondl [10] have studied coupling between a van der Pol oscillator and a damped, linear oscillator, and Linkens [11, 12] has investigated pairs and larger sets of van der Pol oscillators coupled by nonlinear velocity terms. In these studies harmonic balance was generally used. More in the spirit of this paper, Cohen and Neu [13] and Neu [14] have studied pairs of weakly coupled limit cycle oscillators by two-time perturbation techniques. Of course, the externally (periodically) forced van der Pol oscillator has been far more widely studied, notably by van der Pol [15], Cartwright [16], Cartwright and Littlewood [17], Levinson [18] and Levi [19]. See Hayashi [20] or Minorsky [8] for further references. In connection with biological applications, the work of Pavlidis [7] should also be noted.

The plan of this paper is as follows. After discussing the topological structure of the phase space and the existence of invariant tori in Section 2, we derive equations for the slowly varying amplitudes and phases of almost sinusoidal solutions in Section 3. We then turn to the case of diffusive coupling (equation (2)) and study bifurcations (Section 4) and stability (Section 5) of phase locked periodic motions. Finally we describe the manner in which such motions are lost and drifting oscillations arise.

2. TOPOLOGICAL CONSIDERATIONS

For the special case of (1) for which there is no coupling, the states of both oscillators will almost always asymptotically approach their respective stable limit cycles. For small ε the limit cycle of a single van der Pol oscillator is known to be nearly a circle of radius 2 centered at the origin in the $x_i - \dot{x}_i$ phase plane (cf. Minorsky [8]). Since the motions on the respective limit cycles are uncoupled, any phase difference is possible. Once a phase difference is determined (by choice of the initial conditions) it stays constant in time since both oscillators have the same period.

For sufficiently small coupling it may be expected that there will exist periodic motions in which both oscillators remain close to their uncoupled limit cycles. Now, however, the phases of the two oscillators may be "locked", i.e. the difference in phase angles may approch a certain constant value regardless of initial conditions.

Although the phase space for system (1) is four dimensional ($=\mathbb{R}^4$), an approximate picture of the motion may be obtained in fewer dimensions if the coupling is small. To accomplish this we choose as coordinates the (nonconstant) amplitude R_i and (nonconstant) phase angle θ_i of each oscillator ($i = 1, 2$). If we ignore the deviations of R_i from its uncoupled value ($=2$) then the motion may be described by a flow on the torus $T^2 = S^1 \times S^1$ covered by θ_1 and θ_2.

For the uncoupled case this flow is given by

$$\dot{\theta}_1 = -1, \quad \dot{\theta}_2 = -1, \quad R_1, R_2 = 2 + O(\varepsilon); \tag{3}$$

i.e. a uniform translation in the direction $\theta_1 = \theta_2$ with no limit cycles or equilibria; the toroidal phase space is filled with closed orbits, and all solutions except $R_1 = R_2 = 0$ approach this torus exponentially fast as $t \to +\infty$. Since all solutions starting on the torus remain on it for all time, it is called invariant under the flow, or simply *invariant*.

In the case of weak coupling, however, one or more pairs of limit cycles could be expected (in each pair one limit cycle would be stable and one unstable due to the topology of the torus T^2.) See Fig. 1. Since the flow is a small perturbation of the flow (3), the limit cycles would be expected to lie close to the direction $\theta_1 = \theta_2$. Thus a limit cycle L could be approximately described by giving the associated value of the phase difference $\phi = \theta_1 - \theta_2$. This value $\phi = \bar{\phi}$ corresponds to the θ_1-intercept of the straight line L, Fig. 1.

Now suppose that the coupling is small in the following sense: take $0 < \varepsilon \ll 1$ fixed and let the coefficients of the coupling matrices **A** and **B** increase from zero. Since the uncoupled four dimensional system possesses an invariant exponentially attracting torus,

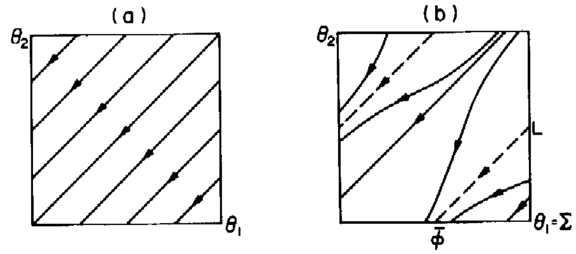

Fig. 1. Flows on the torus: (a) no coupling; (b) weak coupling. In (b), L represents a stable limit cycle (shown dashed) and $\bar{\phi}$ its θ_1-intercept.

this torus will persist for small coupling, perhaps changing its shape and position, but remaining smooth and close to the original torus composed of the two circles $R_1 = R_2 = 2$. The proof of this assertion relies on the theory of normally hyperbolic invariant manifolds (cf. Hirsh–Pugh–Shub [21], Chillingworth [22]) and may be taken as a rigorous justification for the perturbation calculations which follow.

While the invariant torus as a whole is maintained, the structure of orbits within it is not and whether one obtains phase locked periodic motions or drifting oscillations must be determined by other means. This we do below. However, note that the topological structure of the orbits as $t \to +\infty$ is restricted by the requirement that they be asymptotic to the two dimensional torus. Thus we may effectively reduce the four dimensional problem to a two dimensional one, if the coupling is small and we are concerned only with behavior as $t \to \infty$. The coordinates θ_1 and θ_2 still serve to describe the coupled toroidal flow. This is the geometrical picture behind Cohen and Neu's [13, 14] perturbation calculations.

In fact we may further reduce the dimension by considering the Poincaré map P, (cf. Chillingworth [22]) associated with the flow on the torus. Choosing a surface of section $\Sigma = \{(\theta_1, \theta_2) \in T^2 | \theta_2 = 0\}$ the flow then defines our map $P: \Sigma \to \Sigma$ by setting $P(x_0) = x(\tau, x_0)$, where $x(\tau, x_0)$ is the solution curve starting at $x_0 \in \Sigma$ and τ the time for a single circuit, θ_2 going from 0 to 2π. Clearly Σ is a topological circle and may be parameterized by θ_1. A limit cycle L (i.e. a phase-locked oscillation) on T^2 corresponds to a fixed point $\bar{\phi}$ of P. (Fig. 1b), since $x(\tau, x_0) = x_0$ in this case.

This notion of the Poincaré map has proved to be particularly useful in dealing with numerical integrations of system (1). Whereas the usual result of a numerical simulation is four columns of figures representing a phase path in $\mathbb{R}^4$, by converting to R_i, θ_i coordinates, suppressing the R_i coordinates and outputing only those values of θ_1 which correspond to values of $\theta_2 = 0$, a numerical search for periodic motions can be greatly simplified. It is to be noted that although we used the digital computer in this way as an experimental tool, all the results presented in this paper are analytical.

We must stress that, if the coupling is of the same order of magnitude as the nonlinear term **U**, then no invariant torus need exist. Our perturbation scheme applies to this case also, and the results must therefore be interpreted with some care. The existence of an invariant torus relies on the attraction normal to the torus being considerably stronger than motions within the torus, and if this is not satisfied (as it is not when **U**, **A** and **B** are of similar orders) then no torus exists. In the bifurcation calculation of Section 4 below, we do effectively take α, β and Δ small and thus our final equation (30) can be interpreted as an expression governing the phase difference on the torus. We hope to return to the general system, with coupling and nonlinearity of the same order, in subsequent work.

3. APPROXIMATE SOLUTIONS

We will use the two-variable expansion perturbation method [23, 24] to obtain an approximate solution to (1), valid to $O(\varepsilon^2)$. It is to be noted that the same approximate solution can be obtained by other (equivalent) methods such as averaging and harmonic balance.

We replace the independent variable t by two new independent variables, ξ and η:

$$\xi = \Omega t, \quad \Omega = 1 + \Omega_1 \varepsilon + \Omega_2 \varepsilon^2 + \dots \tag{4}$$

$$\eta = \varepsilon t. \tag{5}$$

Here ξ is a stretched time variable while η is a slow time variable. Using the chain rule,

$$\frac{d\mathbf{u}}{dt} = \Omega \frac{\partial \mathbf{u}}{\partial \xi} + \varepsilon \frac{\partial \mathbf{u}}{\partial \eta} \tag{6}$$

$$\frac{d^2\mathbf{u}}{dt^2} = \Omega^2 \frac{\partial^2 \mathbf{u}}{\partial \xi^2} + 2\Omega\varepsilon \frac{\partial^2 \mathbf{u}}{\partial \xi \partial \eta} + \varepsilon^2 \frac{\partial^2 \mathbf{u}}{\partial \eta^2}. \tag{7}$$

Neglecting terms of $O(\varepsilon^2)$, equation (1) becomes

$$(1 + 2\Omega_1 \varepsilon)\mathbf{u}_{\xi\xi} + 2\varepsilon u_{\xi\eta} + \mathbf{u} - \varepsilon(I - U^2)\mathbf{u}_\xi = \varepsilon(\mathbf{A}\mathbf{u} + \mathbf{B}\mathbf{u}_\xi) \tag{8}$$

where subscripts represent partial differentiation.

Next we expand the dependent variables $x_1(\xi, \eta)$ and $x_2(\xi, \eta)$ in power series in ε:

$$x_i(\xi, \eta) = x_{i0}(\xi, \eta) + x_{i1}(\xi, \eta)\varepsilon + O(\varepsilon^2). \tag{9}$$

Substituting (9) into (8), collecting terms and equating to zero coefficients of like powers of ε, we obtain the four equations

$$x_{i0\xi\xi} + x_{i0} = 0, \quad i = 1,2 \tag{10}$$

$$x_{i1\xi\xi} + x_{i1} = (1 - x_{i0}^2)x_{i0\xi} - 2x_{i0\xi\eta} - 2\Omega_1 x_{i0\xi\xi} + \sum_{j=1}^{2} (a_{ij}x_{j0} + b_{ij}x_{j0\xi}), \quad i = 1,2 \tag{11}$$

where a_{ij}, b_{ij} are the elements of the matrices $\mathbf{A}$, $\mathbf{B}$.

Equation (1) have the solution

$$x_{i0} = A_i(\eta) \cos \xi + B_i(\eta) \sin \xi, \quad i = 1,2. \tag{12}$$

We substitute (12) into (11) and require that $x_{i1}(\xi, \eta)$ be uniformly valid for all $\xi > 0$. In order that x_{i1} have no secular (resonance) terms, the coefficients of $\cos \xi$ and $\sin \xi$ on the right-hand side of (11) must vanish, giving the four equations:

$$2A_i' - A_i + 2\Omega_1 B_i + A_i(A_i^2 + B_i^2)/4 + \sum_{j=1}^{2} (a_{ij}B_j - b_{ij}A_j) = 0, \ i = 1, 2 \tag{13}$$

$$-2B_i' + B_i + 2\Omega_1 A_i - B_i(A_i^2 + B_i^2)/4 + \sum_{j=1}^{2} (a_{ij}A_j + b_{ij}B_j) = 0, \quad i = 1, 2 \tag{14}$$

where primes represent differentiation with respect to η.

Next we transform to polar coordinates,

$$x_{i0} = R_i(\eta) \cos (\xi - \theta_i(\eta)), \quad i = 1, 2; \tag{15}$$

i. e.,

$$A_i = R_i \cos \theta_i, \tag{16}$$

$$B_i = R_i \sin \theta_i, \tag{17}$$

and replace (13) and (14) by (13) $A_i - (14) \cdot B_i$ and $(13) \cdot B_i + (14) \cdot A_i$ for $i = 1, 2$ giving four equations for the slowly varying amplitudes, $R_i(\eta)$ and phases, $\theta_i(\eta)$:

$$2R_i R_i' + R_i^2(R_i^2/4 - 1 - b_{ii}) + R_1 R_2[K_i \sin (\theta_1 - \theta_2) + L_i \cos (\theta_1 - \theta_2)] = 0, \quad i = 1, 2, \tag{18}$$

where

$$K_1 = -a_{12}, L_1 = -b_{12},$$

$$K_2 = a_{21}, \quad L_2 = -b_{21};$$

$$-2R_i^2\theta_i' + R_i^2(2\Omega_1 + a_{ii}) + R_1 R_2[M_i \cos (\theta_1 - \theta_2) + N_i \sin (\theta_1 - \theta_2)] = 0, \quad i = 1, 2, \tag{19}$$

where

$$M_1 = a_{12}, \quad N_1 = -b_{12},$$

$$M_2 = a_{21}, \quad N_2 = b_{21}.$$

By dividing (19) by R_i^2 and subtracting, the variables θ_1, θ_2 can be made to appear only in the form $\theta_1 - \theta_2$. This permits us to reduce the problem to a flow on $\mathbb{R}^2 \times \mathbb{S}^1$ with coordinates R_1, R_2 and $\phi = \theta_1 - \theta_2$;

$$2R_1' + R_1(R_1^2/4 - 1 - b_{11}) - R_2(a_{12} \sin \phi + b_{12} \cos \phi) = 0 \tag{20}$$

$$2R_2' + R_2(R_2^2/4 - 1 - b_{22}) + R_1(a_{21} \sin \phi - b_{21} \cos \phi) = 0 \tag{21}$$

$$-2\phi' + (a_{11} - a_{22}) + \left(a_{12}\frac{R_2}{R_1} - a_{21}\frac{R_1}{R_2}\right)\cos \phi - \left(b_{12}\frac{R_2}{R_1} + b_{21}\frac{R_1}{R_2}\right)\sin \phi = 0. \tag{22}$$

Note that while $a_{12}, a_{21}, b_{12}, b_{21}$ represent true coupling coefficients, the coefficients a_{11}, a_{22}, b_{11}, b_{22} represent self-feedback. The term $\varepsilon(a_{11} - a_{22})$ (cf. equation (22)) equals the difference in squares of uncoupled oscillator frequencies. In the absence of true coupling ($a_{12} = a_{21} = b_{12} = b_{21} = 0$) the condition for phase-locking (i.e. $\phi' = 0$) cannot be fullfilled unless $a_{11} = a_{22}$. The terms εb_{11} and εb_{22} are damping coefficients. In the absence of true coupling, the b_{ii}'s influence the size R_i of the uncoupled limit cycles. From (20) and (21,

$$R_i = 2(1 + b_{ii})^{1/2}. \tag{23}$$

In order for the approximate solution (12) to exhibit a phase-locked periodic motion, the flow (20)–(22) must possess an equilibrium solution. By setting $R_1' = R_2' = \phi' = 0$ in (20)–(22) we reduce the problem of finding such solutions to the study of three algebraic equations. The phase difference, ϕ, and the Poincaré map $P: \Sigma \to \Sigma$ are related in the following manner: suppose that a phase locked periodic orbit exists, given (approximately) by $\theta_2 = \theta_1 - \bar{\phi}$ or $\bar{\phi} = \theta_1 - \theta_2$. Since the Poincaré section Σ is taken at $\theta_2 = 0$ this corresponds to a fixed point at $\theta_1 = \bar{\phi}$ ($P(\bar{\phi}) = \bar{\phi}$), cf. Fig. 1.

4. DIFFUSIVE COUPLING

We shall, for simplicity, restrict the rest of the discussion to the special case of two oscillators with diffusion coupling, equation (2). For this choice of **A**, **B**, equation (20)–(22) become:

$$2R_1'/4 - 1 + \beta) - R_2(\alpha \sin \phi + \beta \cos \phi) = 0, \tag{24}$$

$$2R_2' + R_2(R_2^2/4 - 1 + \beta) + R_1(\alpha \sin \phi - \beta \cos \phi) = 0, \tag{25}$$

$$-2\phi' + \Delta + \alpha\left(\frac{R_2}{R_1} - \frac{R_1}{R_2}\right)\cos \phi - \beta\left(\frac{R_2}{R_1} + \frac{R_1}{R_2}\right)\sin \phi = 0 \tag{26}$$

The number of equilibria exhibited by these equations depends upon the values of the parameters α, β, Δ. For example if $\alpha = \beta = 0$ but $\Delta \neq 0$ then (26) shows immediately that there are no equilibria. If however $\Delta = 0$ then (24)–(26) exhibit an in-phase mode, $\phi = 0$, $R_1 = R_2$ and an out-of-phase mode, $\phi = \pi$, $R_1 = R_2$. In the special case $\Delta = \alpha = 0$, $\beta \neq 0$, equation (26) reveals that these are the only equilibria. However for the case $\Delta = \beta = 0$, $\alpha \neq 0$ there are two additional equilibria corresponding to the roots of $\cos \phi = 0$, i.e. $\phi = \pm\pi/2$. Thus in the α–β–Δ parameter space there exists a region which contains systems with no phase-locked periodic motions (including the Δ–axis), a region which contains systems with two such motions (including the β-axis), and a region containing systems having four such motions (including the α-axis). We will find the boundaries of these regions in the neighborhood of the origin in α–β–Δ space, as follows:

When $\alpha = \beta = \Delta = 0$ (no coupling), equations (24)–(26) possess the solutions $R_1 = R_2 = 2$, $\phi =$ any constant. Thus for small α, β, Δ we set

$$R_i = 2 + r_i, \quad i = 1, 2 \tag{27}$$

where $r_i \to 0$ as $\alpha, \beta, \Delta \to 0$. Substituting (27) into (24), (25), assuming $r_i = O(\alpha, \beta)$, and neglecting terms of $O(r_i^2)$ gives the following equilibrium values:

$$r_1 = \alpha \sin \phi + \beta(\cos \phi - 1), \tag{28}$$

$$r_2 = -\alpha \sin \phi + \beta(\cos \phi - 1). \tag{29}$$

Substituting (27)–(29) into (26) gives, to $O(r_i^2)$,

$$f(\phi)= -\Delta+\frac{\alpha^2 \sin 2\phi}{1+\beta(\cos\phi-1)}+2\beta\sin\phi=0.$$

Expanding $f(\phi)$ in a Taylor series about $\beta=0$ and neglecting terms of $O(\alpha^2\beta)$, we find

$$f(\phi)= -\Delta+\alpha^2\sin 2\phi+2\beta\sin\phi=0. \tag{30}$$

Depending upon the values of α, β and Δ, equation (30) has up to four roots.

We first consider three special cases, and we take β, $\Delta>0$ for convenience.

(i)
$$\Delta=0:\ \sin\phi(\alpha^2\cos\phi+\beta)=0 \tag{31}$$

Here we have rewritten (30) slightly. Equation (31) always has the roots $\phi=0$, π, and, if $\alpha^2>\beta$ it has two additional roots at

$$\cos\phi= -\beta/\alpha^2. \tag{32}$$

Thus, when $\Delta=0$, the region of four periodic motions is separated from that with two by the curve $\alpha^2=\beta$.

(ii)
$$\beta=0:\ \sin 2\phi=\Delta/\alpha^2. \tag{33}$$

If $\alpha^2>|\Delta|$ we have four periodic motions, if $\alpha^2=\Delta$ two at $\pi/4$ and $5\pi/4$, (or at $3\pi/4$ and $7\pi/4$ if $\alpha^2=-\Delta$, i.e. if $\Delta<0$), and if $\alpha^2<|\Delta|$ there will be no periodic motions.

(iii)
$$\alpha=0:\ \sin\phi=\Delta/2\beta. \tag{34}$$

If $2\beta>\Delta$ there are two periodic motions, if $2\beta=\Delta$ one [at $\pi/2$] and if $2\beta<\Delta$ there are no periodic motions.

Equations (31)–(34) give partial information on the boundaries or bifurcation sets separating regions in α–β–Δ space with different numbers of periodic motions, the results are summarized in fig. 2. It is also possible to plot bifurcation diagrams; graphs of ϕ versus α (for β, Δ, fixed) for these three cases. These graphs show clearly how the periodic motions appear and disappear as the parameters are varied. The stability types shown in Fig. 2 are derived from the calculations in the next section.

In order to find a general analytical expression for the boundary surfaces or bifurcation sets separating regions containing systems with the same number of phase-locked periodic motions, we note that as one crosses such a boundary two equilibria coalesce. Hence on the boundaries, equation (30) will possess a double root, and we therefore require that

$$f'(\phi)=2\alpha^2\cos 2\phi+2\beta\cos\phi=0. \tag{35}$$

Solving (35) for $\cos\phi$,

$$\cos\phi=-\frac{\beta\pm(\beta^2+8\alpha^4)^{1/2}}{4\alpha^2}. \tag{36}$$

Substituting (36) into(30), we find the following expression for the boundary surfaces:

$$-2\Delta\pm\frac{(3\beta\pm(\beta^2+8\alpha^4)^{1/2})(8\alpha^4-2\beta^2\pm 2\beta(\beta^2+8\alpha^4)^{1/2})^{1/2}}{4\alpha^2}=0. \tag{37}$$

Figure 3(a) shows plots of the level curves of (37) for various values of α taken as constant. These are sections through the two dimensional bifurcation set in α–β–Δ space, see Fig. 3(b).

It is in fact possible to obtain steady state (phase-locked) solutions of equations (24)–(26) without assuming α, β, Δ small and expanding. The three equations can be reduced to a single polynomial equation which can then be solved numerically. We indicate how this can be done in an appendix, but we have not carried out the necessary numerical work, nor have we compared these "exact" solutions with the approximate solutions found above.

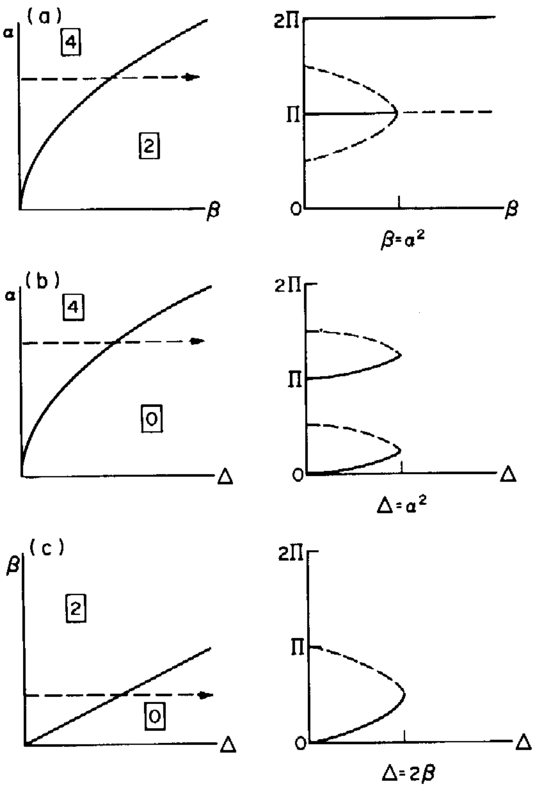

Fig. 2. Bifurcation sets and bifurcation diagrams for three special cases: (a) $\Delta=0$; (b) $\beta=0$; (c) $\alpha=0$. The boxed numerals refer to the number of periodic motions. The bifurcation diagrams represent the evolution of solutions in ϕ space as one follows the dashed arrow path on the parameter plane. Solid lines represent stable phase-locked motions, dashed lines unstable.

5. STABILITY OF PERIODIC MOTIONS

In order to investigate the stability of the phase-locked periodic motions we consider the stability of the equilibria of (24)–(26). We set

$$R_i = R_{i0} + \rho_i, \quad i=1,2 \tag{38}$$

$$\phi = \phi_0 + \psi \tag{39}$$

where R_{i0} and ϕ_0 are coordinates of an equilibrium solution, and where ρ_i and ψ are small perturbations about equilibrium. Substituting (38), (39) into (24)–(26) and linearizing in ρ_i, ψ we obtain:

$$2\mathbf{v}' = \mathbf{M}\mathbf{v} \tag{40}$$

where

$$\mathbf{v} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \psi \end{bmatrix}$$

and where the elements m_{ij} of the matrix $\mathbf{M}$ are given by

$$m_{11} = -(\tfrac{3}{4}R_{10}^2 - 1 + \beta), \qquad m_{12} = \alpha \sin\phi_0 + \beta\cos\phi_0,$$

$$m_{13} = R_{20}(\alpha\cos\phi_0 - \beta\sin\phi_0), \quad m_{21} = -\alpha\sin\phi_0 + \beta\cos\phi_0,$$

$$m_{22} = -(\tfrac{3}{4}R_{20}^2 - 1 + \beta), \qquad m_{23} = R_{10}(-\alpha\cos\phi_0 - \beta\sin\phi_0),$$

$$m_{31} = -\frac{R_{20}}{R_{10}^2}(\alpha\cos\phi_0 - \beta\sin\phi_0) - \frac{1}{R_{20}}(\alpha\cos\phi_0 + \beta\sin\phi_0),$$

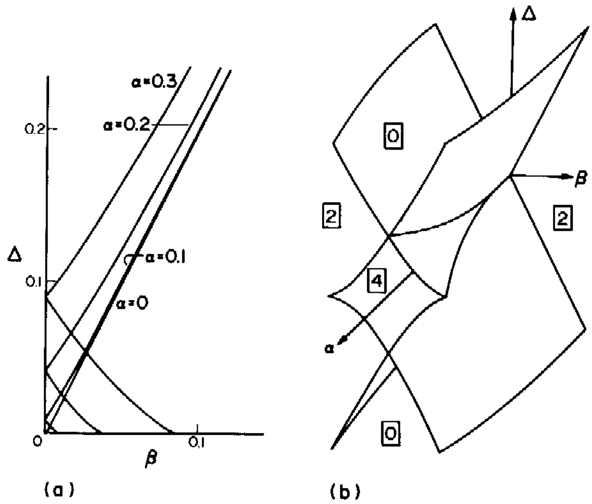

Fig. 3. The bifurcation set, equation (37), valid for small α, β, Δ: (a) level curves for fixed α; (b) the two-dimensional bifurcation set. Boxed numerals refer to the number of periodic motions.

$$m_{32}=\frac{R_{10}}{R_{20}^2}(\alpha\cos\phi_0+\beta\sin\phi_0)+\frac{1}{R_{10}}(\alpha\cos\phi_0-\beta\sin\phi_0),$$

$$m_{33}=-\frac{R_{20}}{R_{10}}(\alpha\sin\phi_0+\beta\cos\phi_0)+\frac{R_{10}}{R_{20}}(\alpha\sin\phi_0-\beta\cos\phi_0).$$

The phase-locked periodic motion characterized by R_{i0} and ϕ_0 will be asymptotically stable if all of the eigenvalues of the matrix **M** have negative real parts, and unstable if any eigenvalue has positive real part.

We will now investigate stability of those periodic motions which lie along the coordinate axes in α–β–Δ space.

Consider first the β-axis, $\Delta=\alpha=0$. The two equilibria for this case satisfy the conditions

$$\phi_0=0,\ \pi \tag{41}$$

whereupon the matrix **M** becomes

$$\mathbf{M}=\begin{bmatrix}-(\frac{3}{4}R_{10}^2-1+\beta) & \pm\beta & 0\\ \pm\beta & -(\frac{3}{4}R_{20}^2-1+\beta) & 0\\ 0 & 0 & \mp\left(\frac{R_{20}}{R_{10}}+\frac{R_{10}}{R_{20}}\right)\end{bmatrix} \tag{42}$$

where the upper (lower) signs correspond to $\phi_0=0$ ($\phi_0=\pi$). For β small, equations (27)–(29) show that

$$R_{10}=R_{20}=2-\beta\pm\beta+0(\beta^2). \tag{43}$$

Substituting (43) into (42), we find that

$$\mathbf{M}=\begin{bmatrix}-2+0(\beta) & 0(\beta) & 0\\ 0(\beta) & -2+0(\beta) & 0\\ 0 & 0 & \mp 2+0(\beta^2)\end{bmatrix} \tag{44}$$

Here the eigenvalues of **M** are $-2+0(\beta)$, $-2+0(\beta)$ and $\mp 2+0(\beta^2)$. Thus for β small the $\phi_0=0$ solution is stable while the $\phi_0=\pi$ solution is unstable. See Fig. 4.

Next consider the α-axis, $\Delta=\beta=0$. The four equilibria for this case satisfy the equations (from (24)–(26))

$$\phi_0=0,\ \pi,\qquad R_{10}=R_{20}=2 \tag{45}$$

or

$$\phi_0 = \pm\pi/2, \quad R_{10}(R_{10}^2/4-1) = \pm\alpha R_{20}$$
$$R_{20}(R_{20}^2/4-1) = \mp\alpha R_{10} \tag{46}$$

where upper (lower) signs correspond to $\phi_0 = \frac{\pi}{2}\left(\phi_0 = -\frac{\pi}{2}\right)$. The matrix $\mathbf{M}$ becomes

$$\mathbf{M} = \begin{bmatrix} -(\frac{3}{4}R_{10}^2-1) & \alpha\sin\phi_0 & R_{20}\alpha\cos\phi_0 \\ -\alpha\sin\phi_0 & -(\frac{3}{4}R_{20}^2-1) & -R_{10}\alpha\cos\phi_0 \\ -\left(\frac{R_{20}}{R_{10}^2}+\frac{1}{R_{20}}\right)\alpha\cos\phi_0 & \left(\frac{R_{10}}{R_{20}^2}+\frac{1}{R_{10}}\right)\alpha\cos\phi_0 & \left(\frac{R_{10}}{R_{20}}-\frac{R_{20}}{R_{10}}\right)\alpha\sin\phi_0 \end{bmatrix} \tag{47}$$

First consider $\phi_0 = 0, \pi$, equation (45). In this case $\mathbf{M}$ becomes

$$\mathbf{M} = \begin{bmatrix} -2 & 0 & \pm 2\alpha \\ 0 & -2 & \mp 2\alpha \\ \mp\alpha & \pm\alpha & 0 \end{bmatrix} \tag{48}$$

where upper (lower) signs correspond to $\phi_0 = 0$ ($\phi_0 = \pi$). Here the eigenvalues of $\mathbf{M}$ are -2, $-1 \pm \sqrt{1-4\alpha^2}$. Thus $\phi_0 = 0, \pi$ are both stable. See Fig. 4.

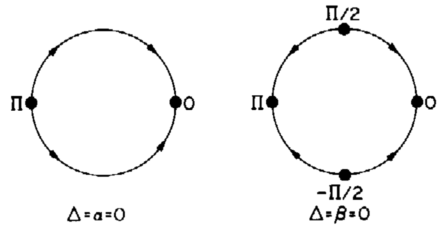

Fig. 4. Stability and location of periodic motions shown on a circle coordinatized by ϕ.

Now consider $\phi_0 = \pm\pi/2$, equation (46). In this case $\mathbf{M}$ becomes

$$\mathbf{M} = \begin{bmatrix} -(\frac{3}{4}R_{10}^2-1) & \pm\alpha & 0 \\ \mp\alpha & -(\frac{3}{4}R_{20}^2-1) & 0 \\ 0 & 0 & \pm\left(\frac{R_{10}}{R_{20}}-\frac{R_{20}}{R_{10}}\right)\alpha \end{bmatrix}. \tag{49}$$

Now from (46), if $\alpha > 0$ and $\phi_0 = \pi/2$, we find that $(R_{10}^2/4-1) > 0$ while $(R_{20}^2/4-1) < 0$, i.e. $R_{10} > 2 > R_{20}$. Hence

$$\frac{R_{10}}{R_{20}} - \frac{R_{20}}{R_{10}} > 0$$

and from (49) $\mathbf{M}$ has a positive eigenvalue, and hence the motion is unstable. Similarly, when $\alpha > 0$ and $\phi_0 = -\pi/2$ we have $R_{20} > 2 > R_{10}$ and again $\mathbf{M}$ has one positive eigenvalue. Also, for $\alpha < 0$ we obtain the same results. These stability conditions are summarized in Fig. 4. In each case the unstable equilibrium point has two negative eigenvalues and one positive eigenvalue. At bifurcation there is a degenerate equilibrium point with two negative eigenvalues and a simple zero eigenvalue.

To obtain more general stability results one can solve (30) numerically, use (27)–(29) to find R_1 and R_2, substitute the resulting values in $\mathbf{M}$ and then find the eigenvalues. However, this requires extensive (though routine) numerical work and in the present case a simpler argument can be given which makes use of the special cases analyzed above to deduce general results, including those used in Fig. 2.

Recall that solutions of (2) for α, β, Δ small approach an invariant torus, T^2, exponentially fast as $t \to +\infty$. All physically interesting motions, including periodic ones, therefore lie on T^2, which can be parameterized by θ_1 and θ_2, the slowly varying phases. Equations (19), (22) (or (26)) show that orbits move continually around T^2 and may thus be characterized only in terms of the phase difference $\phi=\theta_1-\theta_2$. Hence, in describing behavior as $t\to+\infty$, we are essentially dealing with a system defined on a "circle" parameterized by ϕ. (The existence of equations (24) and (25) shows that the circle is not of constant radius.) The flow on the circle provides a continuous approximation of the Poincaré map P: $\Sigma\to\Sigma$.

The topology of the circle implies that stable and unstable fixed points (= phase-locked periodic motions) must alternate, and that their number can only change by bifurcation when two or more coalesce. Coalescence of such a pair, one stable and one unstable, yields a degenerate fixed point at bifurcation, and afterwards, no fixed point at all in the neighborhood. Coalescence of three, unstable, stable and unstable, yields a single unstable fixed point after bifurcation. These situations are shown in Fig. 5 (cf. Figs. (2c) and (2a)). Note that in the first case (Fig. 5(a)) a stable periodic motion vanishes and drifting oscillations, without phase-locking, take over.

These one dimensional bifurcation and stability results can be put into the context of the three dimensional system of (24)–(26) by noting that for small α, β the matrix **M** always has at least *two* negative eigenvalues corresponding to the radial directions of attraction transverse to the circle.

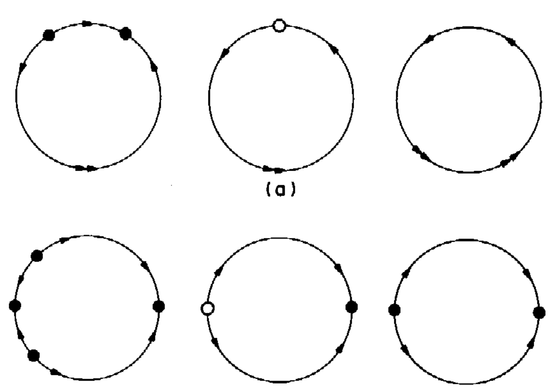

Fig. 5. Coalescence of fixed points (representing phase-locked periodic motions): (a) A pair, cf. Fig. 2(c); (b) three points, cf. Fig. 2(a). The double arrows in (a) indicate fast rate of change of ϕ (cf. discussion of drift oscillations in text). The white dots represent degenerate fixed points which occur at coalescence.

6. THE ONSET OF DRIFT OSCILLATIONS

While a full description of transient behavior is beyond the scope of this paper, it is possible to give a simple description of the loss of phase locking and onset of drift oscillations occurring in a bifurcation such as that of Fig. 5(a). For simplicity we will take $\alpha=0$ and β, Δ small and positive, so that R_1, R_2 can be expanded as in (27). Taking only the lowest order terms in the phase equation (26) we obtain

$$\phi'=\Delta/2-\beta\sin\phi+0(r_i^2). \tag{50}$$

Note that the $0(r_i)$ terms vanish in the expansion of $((R_2/R_1)+(R_1/R_2))$.

Ignoring the terms of $0(r_i^2)$, equation (50) can be easily solved in the two cases $\Delta<2\beta$ and $\Delta>2\beta$:

(i) $\Delta<2\beta$

$$\phi(\eta)=2\arctan\left\{\frac{2}{\Delta}\left[\beta-\mu\coth\left(\frac{\mu\eta}{2}+c_1\right)\right]\right\} \tag{51}$$

where $\mu^2=\beta^2-\Delta^2/4$, and

(ii) $\Delta>2\beta$

$$\phi(\eta)=2\arctan\left\{\frac{2}{\Delta}\left[\beta+\rho\tan\left(\frac{\rho\eta}{2}+c_2\right)\right]\right\} \tag{52}$$

where $\rho^2 = \Delta^2/4 - \beta^2$. In the bifurcation case, $\Delta = 2\beta$, one obtains

$$\phi(\eta) = 2 \arctan(\beta\eta + c_3) - \pi/2. \tag{53}$$

The constants c_i in (51)–(53) may be determined from the initial conditions $\phi(0) = \phi_0$.

Note that, in case (i) as $\eta \to \pm\infty$ all solutions behave as

$$\phi(\eta) \sim 2 \arctan\left(\frac{2(\beta-\mu)}{\Delta}\right), \quad \eta \to +\infty \tag{54a}$$

$$\phi(\eta) \sim 2 \arctan\left(\frac{2(\beta+\mu)}{\Delta}\right), \quad \eta \to -\infty \tag{54b}$$

or, if $\Delta = 2\beta$

$$\phi(\eta) \sim \pi/2 \quad \eta \to \pm\infty. \tag{55}$$

Equations (54) and (55) agree with the fixed points obtained by solving (34) when $\Delta \leqslant 2\beta$.

It is now possible to see, from (52), that when Δ is slightly greater than 2β, so that $0 < \rho \ll 1$, one obtains a very slow periodic modulation of the phase $\phi(\eta)$, since the motion will dwell for a long time near $\phi \approx \pi/2$. Cohen and Neu [13] and Neu [14] noted such a result and related it to observations on "rhythm splitting" in coupled chemical oscillators.

7. CONCLUSIONS

In this paper we have analysed a pair of weakly nonlinear van der Pol oscillators with general weak linear coupling. We have derived differential equations (20)–(22) for the slowly varying amplitudes and phase difference, and have discussed the topological interpretation of this set of three equations in the framework of flows on an invariant torus and the Poincaré map.

Specializing our analysis to the case of diffusive coupling and taking the coupling coefficients α, β and the mutual detuning parameter Δ as small, we have studied the existence, stability and bifurcation of phase-locked periodic motions. Up to four such motions can exist, two stable and two unstable, and since they lie on the surface of an invariant two dimensional torus their behavior under parameter changes is readily understood.

Finally, we have considered some aspects of the loss of entrainment and the onset of drifting oscillations.

APPENDIX EXACT TREATMENT OF (24)–(26) FOR PERIODIC MOTIONS IN THE CASE OF DIFFUSIVE COUPLING

Setting $R_1' = R_2' = \phi' = 0$ in (24)–(26) and taking $(25)\cdot R_1 + (24)\cdot R_2$, $(25)\cdot R_1 - (24)\cdot R_2$ and $(26)\cdot R_1R_2$, we obtain

$$\left[\frac{R_1^2+R_2^2}{4} + 2(\beta-1)\right]R_1R_2 - \alpha \sin\phi(R_2^2 - R_1^2) - \beta\cos\phi(R_1^2+R_2^2) = 0 \tag{A1}$$

$$\left[\frac{R_2^2-R_1^2}{4}\right]R_1R_2 + \alpha\sin\phi(R_1^2+R_2^2) + \beta\cos\phi(R_2^2-R_1^2) = 0 \tag{A2}$$

$$\Delta R_1R_2 + \alpha\cos\phi(R_2^2-R_1^2) - \beta\sin\phi(R_1^2+R_2^2) = 0. \tag{A3}$$

Next taking $(A2)\cdot\alpha - (A3)\cdot\beta$ and $(A2)\cdot\beta + (A3)\cdot\alpha$ we obtain

$$\sin\phi = \left[\frac{4\beta\Delta - \alpha(R_2^2-R_1^2)}{4(\alpha^2+\beta^2)}\right]\left(\frac{R_1R_2}{R_1^2+R_2^2}\right), \tag{A4}$$

$$\cos\phi = -\left[\frac{4\alpha\Delta + \beta(R_2^2-R_1^2)}{4(\alpha^2+\beta^2)}\right]\left(\frac{R_1R_2}{R_2^2-R_1^2}\right). \tag{A5}$$

Now we let

$$R = R_1^2 + R_2^2, \qquad Q = R_2^2 - R_1^2, \tag{A6}$$

and note that

$$R_1^2R_2^2 = \frac{R^2 - Q^2}{4}. \tag{A7}$$

Using the trigonometrical identity $\sin^2\phi + \cos^2\phi = 1$ and equations (A4) and (A5) we obtain

$$\frac{4}{R^2 - Q^2} = \left[\frac{4\beta\Delta - \alpha Q}{4(\alpha^2+\beta^2)R}\right]^2 + \left[\frac{4\alpha\Delta + \beta Q}{4(\alpha^2+\beta^2)Q}\right]^2, \tag{A8}$$

which is the first of our equations relating R and Q. To obtain a second, independent equation we take (A1)$\cdot Q$ + (A2)$\cdot R$ and substitute (A4) for $\sin\phi$:

$$\left[\frac{RQ}{2} + 2Q(\beta - 1)\right] + \frac{\alpha(R^2 - Q^2)(4\beta\Delta - \alpha Q)}{4(\alpha^2+\beta^2)R} = 0. \tag{A9}$$

Rewriting (A9) as a quadratic in R we have

$$[2Q(\alpha^2+\beta^2) + \alpha(4\beta\Delta - \alpha Q)]R^2 + [8Q(\beta-1)(\alpha^2+\beta^2)]R - \alpha Q^2(4\beta\Delta - \alpha Q) = 0 \tag{A10}$$

which may be solved for R in terms of Q giving two roots.

These may then be substituted into (A8) to yield equations in Q alone which can be solved numerically. Once solutions for R and Q are available, equations (A6) and (A4), (A5) may be used to find R_1, R_2 and ϕ.

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Resume

Nous etudions un couple d'oscillateurs de Van der Pol faiblement couples et on examine les ramifications des mouvements periodiques a phase bloquee qui apparaissent lorsqu'on fait varier les coefficients de couplage. On utilise des methodes de perturbation et on discute leur relation avec la structure topologique des solutions dans l'espace des phases a quatre dimensions. Bien que le probleme soit formule pour un couplage lineaire general, on discute plus en detail le cas de dephasage plus de couplage diffusif par le deplacement et la vitesse. On montre que dans ce cas il peut exister jusqu'a quatre mouvements periodiques a phase bloquee.

Zusammenfassung:

Wir untersuchen ein Paar schwach gekoppelter van der Polscher Oszillatoren und behandeln die Abzweigungen phasenfixierter periodischer Bewegungen, die sich ergeben wenn die Kopplungskoeffizienten verändert werden. Perturbationsmethoden werden verwendet und deren Bexiehung zur topologischen Struktur von Lösungen in dem vierdimensionalen Phasenraum wird diskutiert. Während das Problem für allgemeine lineare Kopplung formuliert wird, wird der Fall der Entstimmung und der diffusen Kopplung über Versetzung und Geschwindigkeit mit mehr Einzelheiten diskutiert. Es wird gezeigt, dass in diesem Falle bis zu vier phasenfixierte periodische Lösungen bestehen können.