APPLICATIONS OF COMPUTER ALGEBRA

edited by

Richard Pavelle
Symbolics, Inc.

14

DERIVATION OF THE HOPF BIFURCATION FORMULA USING
LINDSTEDT'S PERTURBATION METHOD AND MACSYMA.

RICHARD H. RAND

Department of Theoretical and Applied Mechanics
Cornell University, Ithaca, New York 14853

ABSTRACT

This paper involves the sudden appearance and growth
of a periodic motion called a limit cycle in an autonomous
system of two nonlinear first order ordinary differential
equations. The bifurcation occurs as a parameter is tuned
so that an equilibrium point goes from a stable focus to an
unstable focus. The resulting limit cycle will generically
occur either i) when the equilibrium is stable (in which
case the limit cycle is unstable), or ii) when the
equilibrium is unstable (in which case the limit cycle is
stable). The Hopf bifurcation formula determines which of
these two cases occurs in a given system, and depends in a
complicated way on the second and third derivatives of the
right-hand sides of the differential equations.

While the Hopf formula itself is well-known to many
users, the usual derivations are complicated and less
accessible. In this paper the Hopf formula is derived in a
straightforward fashion using Lindstedt's well-known
perturbation method in conjunction with MACSYMA.

INTRODUCTION

This paper involves the dynamics of a system of two
ordinary differential equations:

(1) $x' = a(u)\ x + b(u)\ y + f(x,y;u)$
(2) $y' = c(u)\ x + d(u)\ y + g(x,y;u)$

where f and g are strictly nonlinear in x and y (i.e.,
their Taylor expansions about x=y=0 have no constant or

linear terms), and where u is a parameter. Associated with
(1),(2) is the corresponding linearized system:

(3)   x' = a(u) x + b(u) y
(4)   y' = c(u) x + d(u) y

We assume that for u=0 the system (3),(4) has purely
imaginary eigenvalues, i.e. there is a center at the
origin. Moreover, we assume that for u small and negative
(positive), the eigenvalues of (3),(4) have negative
(positive) real parts, i.e. there is a stable (unstable)
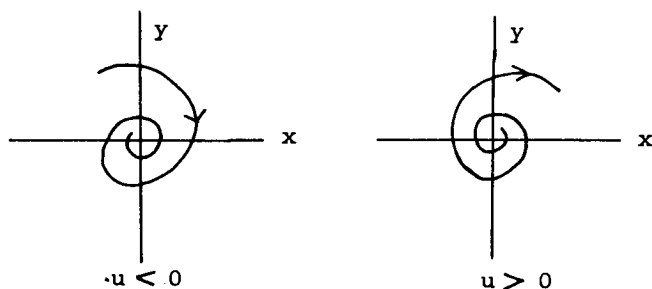focus at the origin.

Fig. 1. Phase portraits for the linearized system.

     In such a case, the nonlinear system (1),(2)
generically (i.e. typically, but not always) undergoes the
birth of a limit cycle, a process called a Hopf bifurcation
(1,2,3). (The birth of a limit cycle can be guaranteed
under certain additional conditions, namely (i) that the
derivative of the real part of the eigenvalues with respect
to u be non-zero at u=0, and (ii) that the origin of
(1),(2) be asymptotically stable or unstable at u=0.)
     There are two generic possibilities, as follows. The
limit cycle may occur for u > 0, in which case it is stable
(the "supercritical" case), or the limit cycle may occur
for u < 0, in which case it is unstable (the "subcritical"
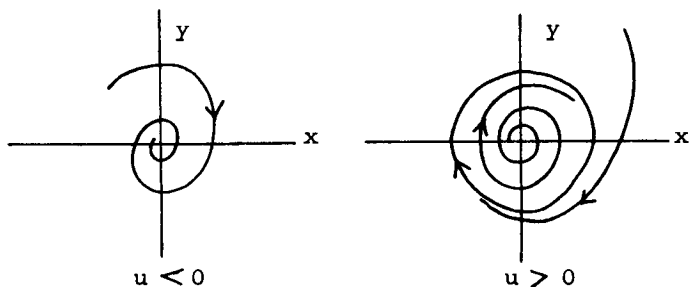case):

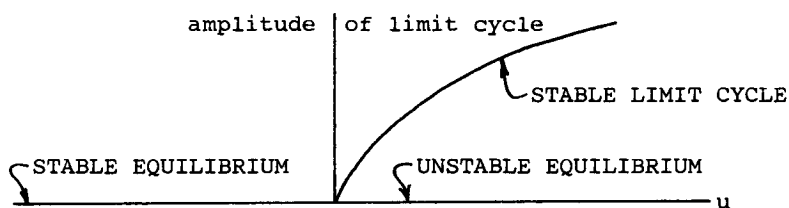Fig. 2.   Phase portraits for the supercritical case.

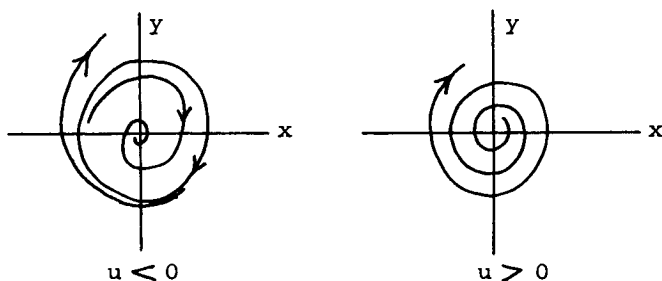Fig. 3.   Bifurcation diagram for the supercritical case.

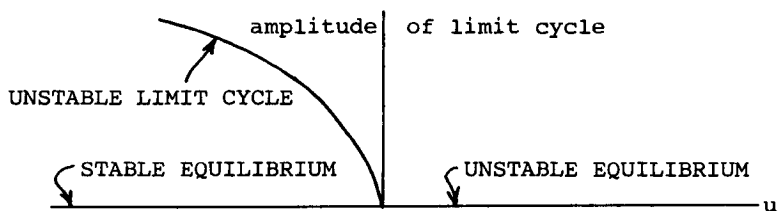Fig. 4.   Phase portraits for the subcritical case.

Fig. 5.   Bifurcation diagram for the subcritical case.

The Hopf bifurcation formula determines which of these two generic cases occurs. The purpose of this paper is to offer a straightforward derivation of this formula using Lindstedt's perturbation method (see (4)) and MACSYMA.

Example: The following equation is a variant of Van der Pol's equation (ref. (4)):

$$(5) \qquad x'' + x - 2 u x' + x^2 x' = 0,$$

When written as a first order system, this becomes:

$$(6) \qquad x' = y$$

$$(7) \qquad y' = -x + 2 u y - x^2 y$$

The eigenvalues of the associated linear system are:

$$(8) \qquad u \pm i (1 - u^2)^{1/2}$$

Note that eq.(8) satisfies the foregoing assumptions, and hence we may expect a Hopf bifurcation. In fact numerical integration reveals there to be a supercritical Hopf bifurcation:

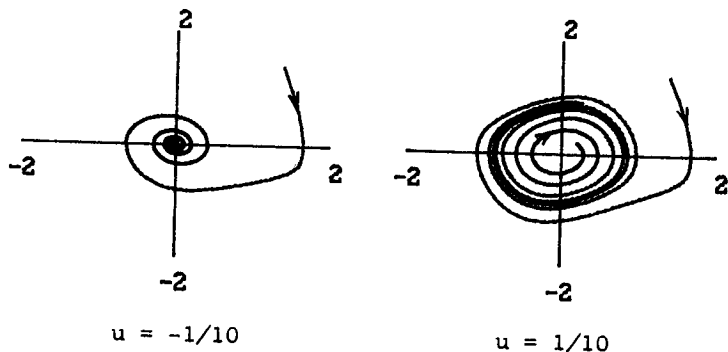u = -1/10   u = 1/10

Fig. 6. Phase portraits obtained by numerical integration of eq.(5).

CANONICAL FORM

We will begin the computation by assuming that the system (1),(2) is already in the following canonical form:

(9) $x' = u\,x - w(u)\,y + f(x,y;u)$

(10) $y' = w(u)\,x + u\,y + g(x,y;u)$

where again f and g are strictly nonlinear in x and y.

Some elementary linear algebra shows that an arbitrary linear system (3),(4) with complex eigenvalues can be transformed to the linearization of (9),(10) by setting:

(11) $xold = b\,xnew$

(12) $yold = (d - a)/2\ xnew - w\,ynew$

where $w = (-(d-a)^2/4 - b\,c)^{1/2}$, and where it turns out that $u = (a+d)/2$.

Example: Continuing the previous example given by eqs.(6),(7), we set

(13) $xold = xnew$

(14) $yold = u\,xnew - w\,ynew$

where $w = (1 - u^2)^{1/2}$. This gives:

(15) $xnew' = u\,xnew - w\,ynew$

(16) $ynew' = w\,xnew + u\,ynew + (u/w)\,xnew^3 - xnew^2\,ynew$

which is of the form (9),(10).

LINDSTEDT'S METHOD

Lindstedt's perturbation method (4) is a well known procedure for obtaining approximate solutions to differential equations which involve a small parameter. We will introduce a small perturbation parameter e into eqs.(9),(10) by scaling variables as follows:

$$(17) \qquad x = e\ X, \quad y = e\ Y, \quad u = e^{2}\ M$$

Next we expand the function $w(u)$ in a power series valid for small u:

$$(18) \qquad w = w(u) = w0 + w1\ u + \ldots$$

$$= w0 + e^{2}\ w1\ M + 0(4)$$

where w0 and w1 are given constants, and where $0(n)$ means terms of order of e raised to the nth power throughout.

We also expand the functions f and g in power series in x and y:

$$(19) \qquad f(x,y,;u) = \frac{fxx}{2}\ x^{2} + \ldots + \frac{fxxx}{6}\ x^{3} + \ldots$$

$$= \frac{fxx}{2}\ e^{2} X^{2} + \ldots + \frac{fxxx}{6}\ e^{3} X^{3} + \ldots + 0(4)$$

and a similar expression for $g(x,y;u)$. Note that the partial derivatives fij are evaluated at x=y=0, but in general will depend on u (see e.g. eq.(16)). However, since u = 0(2), we may take the fij to also be evaluated at u = 0 and still maintain accuracy to 0(4) in eq.(19).

As usual in Lindstedt's method (4), we stretch time t to accomodate the possibility of a dependence of frequency on amplitude in this nonlinear system. We replace t as

independent variable by z, where

(20)
$$z = (w0 + K e^{2} + O(3)) t$$

where K is a constant whose value is to be determined.

Finally we expand the scaled dependent variables X and Y in power series in e:

(21)
$$X(z) = X0(z) + e X1(z) + e^{2} X2(z) + O(3)$$

(22)
$$Y(z) = Y0(z) + e Y1(z) + e^{2} Y2(z) + O(3)$$

We substitute eqs.(17)-(22) into (9),(10), collect terms and equate to zero the coefficient of e raised to the power n, for n=1,2,3,... . This yields a sequence of linear eqs. on Xn(z), Yn(z) which may be solved recursively.

The zero order terms satisfy

(23)
$$X0(z)' = - Y0(z)$$
(24)
$$Y0(z)' = X0(z)$$

which has the solution

(25)
$$X0(z) = A \sin z + B \cos z$$
(26)
$$Y0(z) = B \sin z - A \cos z$$

Since the original problem is autonomous, we may without loss of generality select the initial condition y=0 when t=0, or in the new variables, Y=0 when z=0, which, from (22), gives

(27)
$$Y0(0) = Y1(0) = Y2(0) = ... = 0$$

Eqs.(26),(27) require that A=0 so that

(28) $X0(z) = B \cos z, \quad Y0(z) = B \sin z$

Here the amplitude B, like K in eq.(20), is a constant to be determined.

The MACSYMA session which follows may be outlined thus: We substitute eq.(28) into the differential equations on X1 and Y1 and solve for X1(z), Y1(z). Lindstedt's method generally requires the removal of all resonant (secular, unbounded) terms at each stage of the process, but it turns out that there are no resonant terms in the X1, Y1 equations. Next these results are substituted into the X2, Y2 equations and resonant terms are removed giving two equations for the undetermined coefficients B and K. Solving for the amplitude B and requiring B to be real will yield the Hopf bifurcation formula.

Before beginning, a word about removal of resonant terms in the system:

(29) $x(z)' = - y(z) + a \sin z + b \cos z$
(30) $y(z)' = x(z) + c \sin z + d \cos z$

A particular solution to (29),(30) is

(31) $2 x(z) = (a-d) z \sin z + (b+c) z \cos z + (b-c) \sin z$
(32) $2 y(z) = (b+c) z \sin z + (d-a) z \cos z + (a+d) \sin z$

and therefore for no resonant terms we require

(33) $a - d = 0 \quad \text{and} \quad b + c = 0$

MACSYMA SESSION

Schemes for handling Lindstedt's method in MACSYMA have been extensively treated in (5). (For an introduction to MACSYMA, see (5) or (6).)

We begin by defining the differential equations (9),(10) using stretched time z, eq.(20). Note the use of

variables LCX and LCY to represent lower case x and y
respectively:

(C1) STRETCH:WO+K*E**2;

$$(D1) \qquad\qquad\qquad WO + E^2\ K$$

(C2) 'DIFF(LCX,Z)*STRETCH=U*LCX-W*LCY+F;

$$(D2) \qquad LCX_Z\ (WO + E^2\ K) = -\ LCY\ W + LCX\ U + F$$

(C3) 'DIFF(LCY,Z)*STRETCH=W*LCX+U*LCY+G;

$$(D3) \qquad LCY_Z\ (WO + E^2\ K) = LCX\ W + LCY\ U + G$$

Next we define the functions f and g (cf. eq.(19)):

(C4) F:FXX/2*LCX**2+FXY*LCX*LCY+FYY/2*LCY**2
+FXXX/6*LCX**3+FXXY/2*LCX**2*LCY+FXYY/2*LCX*LCY**2
+FYYY/6*LCY**3;

$$(D4) \quad \frac{FYYY\ LCY^3}{6} + \frac{FXYY\ LCX\ LCY^2}{2} + \frac{FYY\ LCY^2}{2} + \frac{FXXY\ LCX^2\ LCY}{2}$$

$$+ FXY\ LCX\ LCY + \frac{FXXX\ LCX^3}{6} + \frac{FXX\ LCX^2}{2}$$

(C5) G:GXX/2*LCX**2+GXY*LCX*LCY+GYY/2*LCY**2
+GXXX/6*LCX**3+GXXY/2*LCX**2*LCY+GXYY/2*LCX*LCY**2
+GYYY/6*LCY**3;

$$(D5) \quad \frac{GYYY\ LCY^3}{6} + \frac{GXYY\ LCX\ LCY^2}{2} + \frac{GYY\ LCY^2}{2} + \frac{GXXY\ LCX^2\ LCY}{2}$$

$$+ GXY\ LCX\ LCY + \frac{GXXX\ LCX^3}{6} + \frac{GXX\ LCX^2}{2}$$

We complete the specification of the perturbation scheme by
scaling u and w, and then expanding the scaled variables X
and Y (see eqs.(21),(22)):

(C6) U:E**2*M;

$$(D6) \qquad\qquad\qquad E^2\ M$$

```
(C7) W:WO+W1*U;
```

$$(D7) \qquad\qquad E^2 \ M \ W1 \ + \ WO$$

```
(C8) LCX:E*(XO(Z)+E*X1(Z)+E**2*X2(Z));
```

$$(D8) \qquad\qquad E \ (E^2 \ X2(Z) \ + \ E \ X1(Z) \ + \ XO(Z))$$

```
(C9) LCY:E*(YO(Z)+E*Y1(Z)+E**2*Y2(Z));
```

$$(D9) \qquad\qquad E \ (E^2 \ Y2(Z) \ + \ E \ Y1(Z) \ + \ YO(Z))$$

Now all the previous expansions are substituted into the differential equations labeled D2 and D3:

```
(C10) [D2,D3],DIFF$
```

Here and elsewhere we use the $ terminator to supress the display of the resulting expression. Next we Taylor expand and collect terms:

```
(C11) TAYLOR(EV(%),E,0,3)$
```

```
(C12) FOR I THRU 3 DO EQ[I-1]:COEFF(%,E,I)$
```

As a check we display the zeroth order equations (cf. eqs.(23),(24)):

```
(C13) EQ[0]/WO;
```
$$(D13)/R/ \qquad [XO(Z)_Z = - \ YO(Z), \ YO(Z)_Z = XO(Z)]$$

As usual in Lindstedt's method, the first order equations are nonhomogenous versions of the zeroth order:

```
(C14) EQ[1];
```

$$(D14)/R/ \ [WO \ X1(Z)_Z = - \ (2 \ Y1(Z) \ WO - FXX \ XO^2(Z)$$

$$- \ 2 \ FXY \ YO(Z) \ XO(Z) - FYY \ YO^2(Z))/2,$$

$$Y1(Z)_Z \ WO = (2 \ X1(Z) \ WO + GXX \ XO^2(Z) + 2 \ GXY \ YO(Z) \ XO(Z)$$

$$+ \ GYY \ YO^2(Z))/2]$$

We define the zeroth order solution, eq.(28), and
substitute into the first order equations:

```
(C15) [XO(Z)=B*COS(Z),YO(Z)=B*SIN(Z)];
(D15)      [XO(Z) = B COS(Z), YO(Z) = B SIN(Z)]
```

```
(C16) EQ[1],%,DIFF$
```

We clean up the trig terms with TRIGREDUCE before
attempting to solve the first order equations:

```
(C17) EXPAND(TRIGREDUCE(EXPAND(%)));
```

$$
(D17) \quad [WO\ X1(Z) = \frac{B^2\ FXY\ SIN(2\ Z)}{2} - \frac{B^2\ FYY\ COS(2\ Z)}{4}
$$

$$
+ \frac{B^2\ FXX\ COS(2\ Z)}{4} - WO\ Y1(Z) + \frac{B^2\ FYY}{4} + \frac{B^2\ FXX}{4},
$$

$$
WO\ Y1(Z) = \frac{B^2\ GXY\ SIN(2\ Z)}{2} - \frac{B^2\ GYY\ COS(2\ Z)}{4}
$$

$$
+ \frac{B^2\ GXX\ COS(2\ Z)}{4} + WO\ X1(Z) + \frac{B^2\ GYY}{4} + \frac{B^2\ GXX}{4}]
$$

The first order equations are solved using DESOLVE. The
MACSYMA function ATVALUE is used to specify the initial
condition eq.(27):

```
(C18) LOAD([DESOLN,MACSYMA,SHARE])$
```

```
(C19) ATVALUE(Y1(Z),Z=0,0)$
```

```
(C20) DESOLVE(D17,[X1(Z),Y1(Z)]);
```

$$
(D20) \quad [X1(Z) = \frac{(B^2\ GXY - B^2\ FYY + B^2\ FXX)\ SIN(2\ Z)}{6\ WO}
$$

$$
- \frac{(B^2\ GYY - B^2\ GXX + 4\ B^2\ FXY)\ COS(2\ Z)}{12\ WO}
$$

$$- \frac{(2 B^2 GXY - 2 B^2 FYY - B^2 FXX) SIN(Z)}{6 WO}$$

$$+ \frac{(6 X1(0) WO + 2 B^2 GYY + B^2 GXX + 2 B^2 FXY) COS(Z)}{6 WO}$$

$$- \frac{B^2 GYY + B^2 GXX}{4 WO},$$

$$Y1(Z) = - \frac{(B^2 GYY - B^2 GXX + B^2 FXY) SIN(2 Z)}{6 WO}$$

$$- \frac{(4 B^2 GXY - B^2 FYY + B^2 FXX) COS(2 Z)}{12 WO}$$

$$+ \frac{(6 X1(0) WO + 2 B^2 GYY + B^2 GXX + 2 B^2 FXY) SIN(Z)}{6 WO}$$

$$+ \frac{(2 B^2 GXY - 2 B^2 FYY - B^2 FXX) COS(Z)}{6 WO} + \frac{B^2 FYY + B^2 FXX}{4 WO}]$$

Next the zeroth and first order solutions are substituted
into the second order equations, and TRIGREDUCE is again
used to tidy up:

(C21) EQ[2],D15,%,DIFF$

(C22) EXPAND(TRIGREDUCE(EXPAND(%)))$

Finally we isolate the coefficients of sin z and cos z in
order to remove the resonant terms in the second order
equations:

(C23) COEFF(D22,SIN(Z));

$$(D23) \; [- B K = - B M W1 - \frac{7 B^3 FXY GYY}{24 WO} + \frac{B^3 FYY GXY}{6 WO}$$

$$+ \frac{B^3\ FXX\ GXY}{12\ WO} - \frac{5\ B^3\ FXY\ GXX}{24\ WO} + \frac{5\ B^3\ FYY}{24\ WO} + \frac{5\ B^3\ FXX\ FYY}{24\ WO}$$

$$+ \frac{B^3\ FXY^2}{12\ WO} + \frac{B^3\ FXX^2}{12\ WO} + \frac{B^3\ FYYY}{8} + \frac{B^3\ FXXY}{8},$$

$$0 = - \frac{B^3\ GXY\ GYY}{8\ WO} + \frac{5\ B^3\ FYY\ GYY}{24\ WO} + \frac{7\ B^3\ FXX\ GYY}{24\ WO} - \frac{B^3\ GXX\ GXY}{8\ WO}$$

$$+ \frac{B^3\ FXY\ GXY}{12\ WO} - \frac{B^3\ FYY\ GXX}{12\ WO} + \frac{B^3\ FXX\ GXX}{12\ WO} + B\ M + \frac{B^3\ GYYY}{8}$$

$$+ \frac{B^3\ GXXY}{8}]$$

(C24) COEFF(D22,COS(Z));

$$(D24)\ [0 = - \frac{B^3\ FYY\ GYY}{12\ WO} - \frac{7\ B^3\ FXX\ GYY}{24\ WO} - \frac{B^3\ FXY\ GXY}{12\ WO}$$

$$+ \frac{B^3\ FYY\ GXX}{12\ WO} - \frac{5\ B^3\ FXX\ GXX}{24\ WO} + \frac{B^3\ FXY\ FYY}{8\ WO} + \frac{B^3\ FXX\ FXY}{8\ WO}$$

$$+ B\ M + \frac{B^3\ FXYY}{8} + \frac{B^3\ FXXX}{8},$$

$$B\ K = B\ M\ W1 - \frac{B^3\ GYY^2}{12\ WO} - \frac{5\ B^3\ GXX\ GYY}{24\ WO} - \frac{B^3\ FXY\ GYY}{12\ WO} - \frac{B^3\ GXY^2}{12\ WO}$$

$$+ \frac{5\ B^3\ FYY\ GXY}{24\ WO} + \frac{7\ B^3\ FXX\ GXY}{24\ WO} - \frac{5\ B^3\ GXX^2}{24\ WO} - \frac{B^3\ FXY\ GXX}{6\ WO}$$

$$+ \frac{B^3\ GXYY}{8} + \frac{B^3\ GXXX}{8}]$$

Comparison of expressions D23 and D24 with eqs.(29),(30) shows that D23 is of the form [a,c] while D24 represents [b,d]. The conditions (33) for removal of resonant terms involve two equations for B and K. However since we are only interested in B for the Hopf formula, we form only one of the conditions (33), namely b+c = 0:

(C25) PART(D24,1)+PART(D23,2)$

(C26) SOLVE(%,B);
(D26) [B = - 4 SQRT(- M WO/(GYYY WO + GXXY WO + FXYY WO

+ FXXX WO - GXY GYY + FYY GYY - GXX GXY - FXX GXX

+ FXY FYY + FXX FXY)),

B = 4 SQRT(- M WO/(GYYY WO + GXXY WO + FXYY WO + FXXX WO

- GXY GYY + FYY GYY - GXX GXY - FXX GXX + FXY FYY

+ FXX FXY)),

B = O]

SOLVE returns three values for B. We choose the positive value:

(C27) PART(%,2);
(D27) B = 4 SQRT(- M WO/(GYYY WO + GXXY WO + FXYY WO

+ FXXX WO - GXY GYY + FYY GYY - GXX GXY - FXX GXX

+ FXY FYY + FXX FXY))

The result of the preceding calculation may be expressed thus: For small e, the amplitude of the limit cycle is approximately given by

(34) $\text{Amplitude} = e\,B = 4\,(-\,u\,w0\,/\,S\,)^{1/2}$

where the quantity S is defined by the formula

(35) $S = w0\,(\,gyyy + gxxy + fxyy + fxxx\,)$
$$- gxy\,gyy + fyy\,gyy - gxx\,gxy$$
$$- fxx\,gxx + fxy\,fyy + fxx\,fxy$$

in which all the partial derivatives are evaluated when $x=y=u=0$.

In order for the limit cycle to exist, the amplitude (34) must be real. Thus if $S > 0$, the limit cycle occurs when $u < 0$ (the subcritical case), while if $S < 0$, the limit cycle occurs when $u > 0$ (the supercritical case). If $S = 0$, no conclusion may be drawn (the nongeneric case).

Example: For the Van der Pol example of eqs.(15),(16), we find

$$(36) \qquad f = 0, \qquad g = (u/w) x^3 - x^2 y$$

where $w = (1 - u^2)^{1/2}$. The only nonvanishing derivative at $x=y=u=0$ is

$$(37) \qquad gxxy = -2$$

We also see that (cf. eq.(18))

$$(38) \qquad w0 = 1$$

Eqs.(35),(37),(38) give

$$(39) \qquad S = -2 < 0$$

Thus the Hopf theory predicts that we have a supercritical bifurcation in which a stable limit cycle emerges for $u > 0$. The approximate expression (34) for the limit cycle's amplitude becomes

$$(40) \qquad \text{Amplitude} = (8 u)^{1/2}$$

valid for small u. For $u = 1/10$, eq.(40) predicts an amplitude of about 0.89, which approximately agrees with the result of numerical integration shown in Fig. 6.

CONCLUSION

This work involves the use of computer algebra to derive a formula for which other derivations have been given (1,2). All such derivations include a vexatious quantity of algebra, making the use of computer algebra more attractive.

This kind of application of computer algebra is distinctly different from traditional computations in which one seeks the answer to a particular problem. Rather, here we see the computer algebra system as functioning as a theorem-prover. We expect to see the increasing appearance of computer algebra proofs and derivations replacing traditional tedious hand calculations in courses in mathematics and engineering.

ACKNOWLEDGEMENT

REFERENCES

1. Marsden, J.E. and McCracken, M., The Hopf Bifurcation and Its Applications, Springer-Verlag (1976)
2. Guckenheimer, J. and Holmes, P., Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, Springer-Verlag (1983)
3. Hassard, B.D., Kazarinoff, N.D. and Wan, Y.H., Theory and Applications of Hopf Bifurcation, Cambridge University Press (1981)
4. Stoker, J.J., Nonlinear Vibrations, Interscience Publishers (1950)
5. Rand, R.H., Computer Algebra in Applied Mathematics: An Introduction to MACSYMA, Pitman Publishing (1984)
6. MACSYMA Reference Manual, 2 vols., The Mathlab Group, Laboratory for Computer Science, MIT, 545 Technology Square, Cambridge, MA 02139, Version 10 (1983)