Proceedings of the
American Control Conference
Seattle, Washington • June 1995

LIMITED TORQUE SPINUP OF AN UNBALANCED ROTOR ON AN ELASTIC SUPPORT

R. J. Kinsey*
The Aerospace Corporation
El Segundo, California

D. L. Mingori**
University of California, Los Angeles
Los Angeles, California

R. H. Rand†
Cornell University
Ithaca, New York

Abstract

Consider an oscillator with an embedded unbalanced rotor which starts from rest and spins up under the influence of a motor with limited torque. For the case of a constant torque which is small relative to the size of the unbalance, the rotor speed will stall near the resonant frequency of the oscillator—this is referred to as "capture." For larger constant torque, the rotor will spin up through resonance without stalling—this is referred to as "pass-through." For the constant torque capture case, we develop a feedback control strategy which exploits unstable dynamic behavior near resonance in order to drive the system to a state favorable for pass-through—a state which cannot be reached using constant torque. Numerical verification shows that using this feedback strategy as opposed to constant torque, the motor needs approximately one tenth of the torque capability to achieve marginal pass-through.

I. Introduction

Consider a spring-mass oscillator with an embedded unbalanced rotor, which we refer to as an unbalanced oscillator. An idealized unbalanced oscillator is shown in Fig. 1. A support base is connected to a wall by a linear spring with spring constant k. Embedded in the support base is a rotor which is statically balanced and which has moment of inertia I. The mass of the support together with the embedded rotor is M. An additional point mass m is attached to the rotor a distance e off the axis of rotation. A right handed coordinate frame with origin at O is defined by the unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$. The support is free to move parallel to $\hat{i}$; its displacement relative to the position in which the spring is unstretched is x. The rotor's axis of rotation is parallel to $\hat{k}$. The angle of rotation

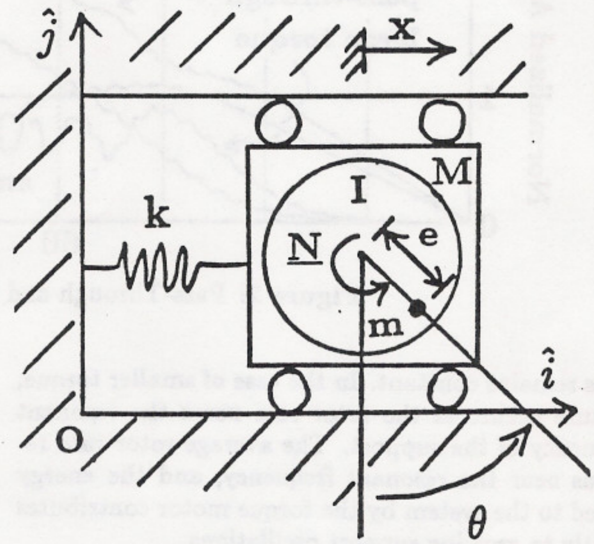

Figure 1: An Idealized Unbalanced Oscillator

θ is measured relative to $-\hat{j}$. The system starts from rest, and the rotor and attached point mass spin up under the action of a motor which provides a torque N along the axis of rotation (parallel to $\hat{k}$). Gravity is perpendicular to the plane of motion and thus does not affect the motion. No energy dissipation occurs anywhere in the system.

During spinup of the embedded rotor, if the size of the rotor unbalance is sufficiently small relative to the motor torque capability, "pass-through" occurs, i.e., the rotor rate passes through the resonant frequency of the support without stalling. On the other hand, if the size of the unbalance is large relative to the motor torque capability, then "capture" occurs, i.e., the rotor rate stalls near the resonant frequency. Figure 2 depicts the rotor rate and the amplitude of support oscillations during pass-through and capture. In the larger torque pass-through case, the rotor rate increases nearly linearly as the rotor spins up through the resonant frequency of the support. While the rotor rate is near the resonant frequency, the amplitude of support oscillations grows. Away from the resonant frequency, the average amplitude of support oscilla-

*Engineering Specialist; Member AIAA
Send correspondence to Dr. R. J. Kinsey
Mail Station: M4/974
The Aerospace Corporation, P.O. Box 92957
Los Angeles, CA 90009
**Professor, Mechanical, Aerospace, and Nuclear Engineering; Associate Fellow AIAA
†Professor, Theoretical and Applied Mechanics

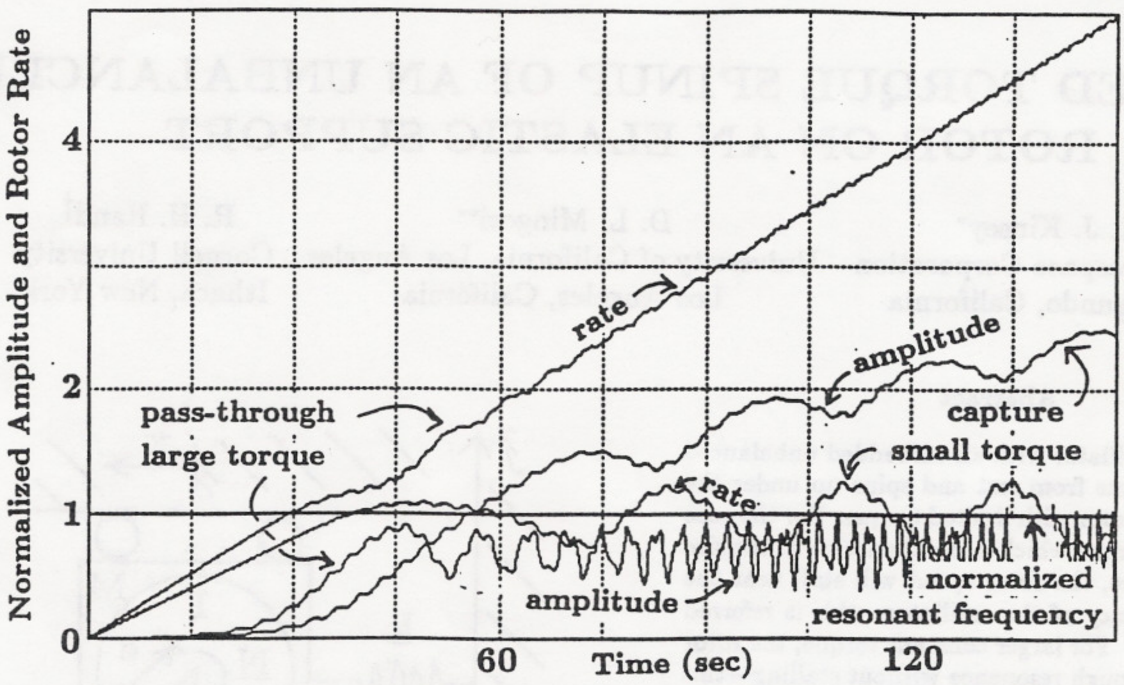

Figure 2: Pass-Through and Capture in an Unbalanced Oscillator

tions remains constant. In the case of smaller torque, capture occurs as the rotor rate nears the resonant frequency of the support. The average rotor rate remains near the resonant frequency, and the energy added to the system by the torque motor contributes mostly to growing support oscillations.

A number of authors have investigated the dynamics of an unbalanced oscillator during rotor spinup. Lewis analyzed an unbalanced oscillator subject to the constraint that rotor acceleration is constant (i.e., rotor rate increases linearly) during spinup [1]. Thus, he considered only pass-through cases. He developed a method for hand calculation of a bound on the maximum amplitude of support oscillations during spinup. Qazi and MacFarlane consider essentially the same system [2]. By means of an analog computer, they demonstrate a feedback control system, involving the precise timing of a step increase in the rotor torque, which reduces the maximum amplitude of support oscillations during passage through resonance. Yee considers capture as well as pass-through, i.e., he does not assume constant (or even positive) rotor acceleration during spinup [3]. Through numerical integration of the exact equations of motion, he quantifies the minimum constant motor torque, as a function of unbalance size, for which pass-through is achieved.

Sanders and Verhulst consider the local behavior of an unbalanced oscillator near resonance [4]. They assume that the torque motor has a linear torque-speed characteristic. Through the method of averaging and numerical integration, they obtain times of passage through resonance as a function of initial conditions. Rand, Kinsey, and Mingori also analyze the local behavior of an unbalanced oscillator near resonance [5]. They obtain an approximate solution based on Jacobian elliptic functions. Then, through variation of parameters and the method of averaging, they derive a bound for the region of initial conditions which lead to capture.

This paper focuses on the development of a closed loop feedback control strategy which achieves pass-through where small constant torque results in capture.

II. Equations of Motion

Using $s\theta$ and $c\theta$ to denote sine and cosine of θ, the governing equations for the unbalanced oscillator may be written as

$$(\mathrm{M}+\mathrm{m})\ddot{\mathrm{x}}+\mathrm{me}(\ddot{\theta}c\theta-\dot{\theta}^2 s\theta)+\mathrm{kx} = 0 \quad (1)$$

$$(\mathrm{I}+\mathrm{me}^2)\ddot{\theta}+\mathrm{me}\ddot{\mathrm{x}}c\theta = \mathrm{N} \quad (2)$$

Transform (1) and (2) into dimensionless form using the following scalings and parameter definitions:

$$\left.\begin{aligned} \mathrm{x} &= z\sqrt{\frac{\mathrm{I}+\mathrm{me}^2}{\mathrm{M}+\mathrm{m}}} \qquad (\;)' = \frac{d}{d\tau} \\ \omega^2 &= \frac{\mathrm{k}}{\mathrm{M}+\mathrm{m}} \qquad \mathrm{L} = \frac{\mathrm{N}}{\omega^2(\mathrm{I}+\mathrm{me}^2)} \\ \tau &= \omega t \quad \varepsilon = \frac{\mathrm{me}}{\sqrt{(\mathrm{I}+\mathrm{me}^2)(\mathrm{M}+\mathrm{m})}} \end{aligned}\right\} \quad (3)$$

The dimensionless equations are

$$z'' + z = -\varepsilon \frac{d^2(s\theta)}{d\tau^2} \qquad (4)$$

$$\theta'' = L - \varepsilon z'' c\theta \qquad (5)$$

In (4) and (5), ε is a dimensionless parameter representing the relative size of the unbalance due to the point mass m, and L is a dimensionless parameter representing the relative size of the motor torque applied to the rotor.

To obtain a useful alternative form of the governing equations, introduce $\overline{z}$ defined as

$$\overline{z} = z + \varepsilon s\theta \qquad (6)$$

In terms of $\overline{z}$ and θ, (4) and (5) become

$$\overline{z}'' + \overline{z} = \varepsilon s\theta \qquad (7)$$

$$\theta'' = \frac{L + \varepsilon \overline{z} c\theta - \varepsilon^2(1+\theta'^2) s\theta c\theta}{1 - \varepsilon^2 c^2\theta} \qquad (8)$$

In order to express (7) in first order form, introduce the transformation

$$u = \overline{z} c\theta - \overline{z}' s\theta \qquad (9)$$

$$v = \overline{z}' c\theta + \overline{z} s\theta \qquad (10)$$

Apply (9) and (10) to (7) and (8) to obtain

$$u' = v(1-\theta') - \varepsilon s^2\theta \qquad (11)$$

$$v' = -u(1-\theta') + \varepsilon s\theta c\theta \qquad (12)$$

$$\theta'' = \frac{L + \varepsilon(uc^2\theta + vs\theta c\theta) - \varepsilon^2(1+\theta'^2) s\theta c\theta}{1-\varepsilon^2 c^2\theta} \qquad (13)$$

The "at rest" initial conditions are

$$u(0) = v(0) = \theta(0) = \theta'(0) = 0 \qquad (14)$$

Eqs. (11) – (13) together with initial conditions (14) form a complete set of dimensionless exact equations of motion for the unbalanced oscillator.

III. Averaged Equations

The function $f(\varepsilon)$ is said to be $O(g(\varepsilon))$ if there exist positive constants M and ε_o such that

$$|f(\varepsilon)| \leq M\,|g(\varepsilon)| \text{ for all } |\varepsilon| \leq \varepsilon_o$$

In Eqs. (11) – (14), ε and L are dimensionless parameters. L represents the normalized motor torque, and ε represents the normalized unbalance. In the following derivation, we assume $\varepsilon \ll 1$, and we neglect terms of $O(\varepsilon^2)$. Since we are interested in systems for which torque is limited, we also assume that $L = \varepsilon K$, where $K = O(1)$ is a constant. Under these assumptions, Eq. (13) becomes

$$\theta'' = \varepsilon K + \varepsilon(u\,c^2\theta + v\,s\theta c\theta) + O(\varepsilon^2) \qquad (15)$$

Resonance occurs when the rotor rate, θ', equals 1, the normalized value of the natural frequency of the support. Now, let $w = \theta' - 1$, so that $w' = \theta''$. We have from (15) that θ' (and therefore w) is slowly varying. We have from (11) and (12) that u and v are quickly varying. The variable θ is also quickly varying near resonance (near $\theta' = 1$). The *method of averaging* [4] can be used to simplify (11), (12), and (15). The method of averaging is applied here via a near-identity transformation. Let

$$\theta = \overline{\theta} - \frac{\varepsilon}{2(2+\overline{w})^2}\left[\overline{u}\,c2\overline{\theta} + \overline{v}\,s2\overline{\theta}\right] \qquad (16)$$

$$w = \overline{w} + \frac{\varepsilon}{2(2+\overline{w})}\left[\overline{u}\,s2\overline{\theta} - \overline{v}\,c2\overline{\theta}\right] \qquad (17)$$

$$u = \overline{u} + \frac{\varepsilon}{2(2+\overline{w})^2}\left[\overline{u}\overline{v}\,c2\overline{\theta} + (2+\overline{w}+\overline{v}^2)s2\overline{\theta}\right] \qquad (18)$$

$$v = \overline{v} - \frac{\varepsilon}{2(2+\overline{w})^2}\left[\overline{u}\overline{v}\,s2\overline{\theta} + (2+\overline{w}+\overline{u}^2)c2\overline{\theta}\right] \qquad (19)$$

The averaged variables are $\overline{\theta}$, $\overline{w}$, $\overline{u}$, and $\overline{v}$. Differentiate (16) - (19) and substitute from (11), (12) and (15). Neglecting terms of $O(\varepsilon^2)$, the resulting equations are

$$\overline{u}' = -\overline{w}\,\overline{v} - \frac{\varepsilon}{2} \qquad (20)$$

$$\overline{v}' = \overline{w}\,\overline{u} \qquad (21)$$

$$\overline{w}' = \varepsilon\left(K + \frac{\overline{u}}{2}\right) \qquad (22)$$

Substitute initial conditions (14) into (16) – (19) to obtain

$$\overline{u}(0) = 0 \qquad (23)$$

$$\overline{v}(0) = \frac{\varepsilon}{2} \qquad (24)$$

$$\overline{w}(0) = -1 \qquad (25)$$

Note that $\overline{\theta}$ does not appear in (20) – (25). It has been "averaged out" by the near-identity transformations. Eqs. (20) – (22) together with initial conditions (23) – (25) form a complete set of averaged equations of motion for the unbalanced oscillator, where the system starts from rest and spins up under the influence of a constant torque.

The averaged equations depend on only the two (dimensionless) parameters ε and K. Hence, for each ε there is one particular value of K for which pass-through is just barely achieved. We call this value K_* and refer to it as the marginal pass-through value. Reference [6] contains a derivation of the following simple formula:

$$K_* = 0.81914\,\varepsilon^{1/3} \qquad (26)$$

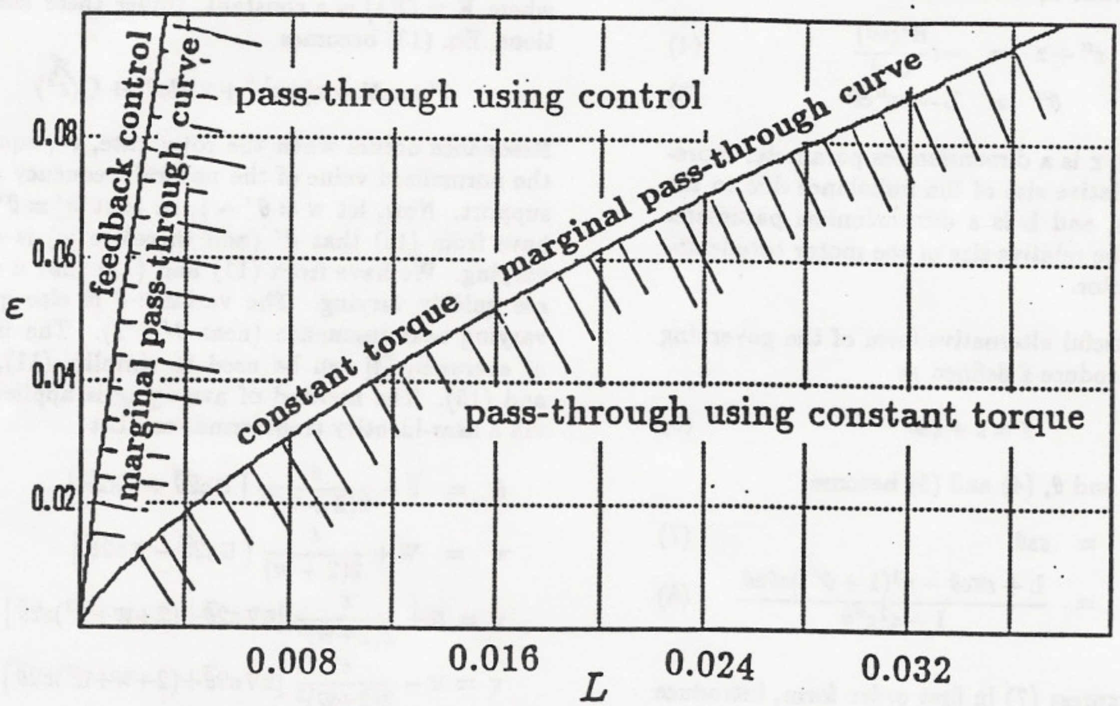

Figure 3: Pass-Through and Capture Regions in the L — $\mathcal{E}$ Plane

The above was used to generate the constant torque marginal pass-through curve shown in Fig. 3.

One might ask whether pass-through can be achieved using constant K < K_*, starting at some point in the resonance plane at $\overline{w} = 0$, instead of starting from rest. As seen in Figure 4, the answer is yes. The figure was generated through extensive numerical integration (460,800 computer simulation runs) of the exact equations of motion [6]. It depicts a region of initial conditions (u(0), v(0)) in the resonance plane for which pass-through occurs even with very small constant torque. We refer to this region as a "hole" in the resonance plane. A similar exercise using the averaged equations generated a slightly larger hole.

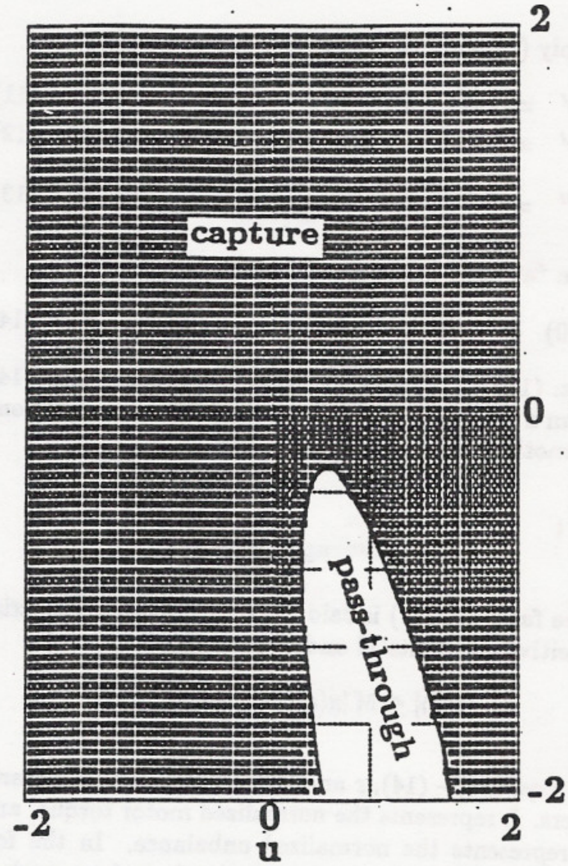

Figure 4: "Hole in the Resonance Plane"

IV. Control Strategy

We assume that the rotor angle and rate, and the support position and velocity are all measured and available for feedback. Under these assumptions, the states $\overline{u}$, $\overline{v}$, $\overline{w}$ in the averaged equations for the oscillator can be calculated from measured variables. We neglect all processing delays and assume that the torque motor has an ideal (unity) transfer function.

Our control strategy is described below. An example of system behavior using this strategy is provided in Figure 5. The figure depicts results from numerical integration of the exact equations (11) – (14), for a

case in which $\varepsilon = 0.1$, $K_* = 0.38$, and $K_{max} = 0.05$, where K_{max} corresponds to the maximum achievable torque magnitude.

Beginning from the (at rest) initial conditions (23) – (25), apply the maximum possible constant torque $L = \varepsilon K_{max}$. Continue using constant torque until $\sqrt{u^2 + v^2}$ first exceeds 0.7, i.e., until large amplitude support motion occurs which indicates that the system is near resonance. Then switch from constant torque to feedback control.

To develop a feedback control law, first apply the polar coordinate transformation

$$\bar{u} = r\sin(\chi) \qquad (27)$$

$$\bar{v} = r\cos(\chi) \qquad (28)$$

to the averaged equations (20) – (22). This gives

$$r' = -\frac{\varepsilon}{2}\sin(\chi) \qquad (29)$$

$$\chi' = -\bar{w} - \frac{\varepsilon}{2r}\cos(\chi) \qquad (30)$$

$$\bar{w}' = \varepsilon K + \frac{\varepsilon r}{2}\sin(\chi) \qquad (31)$$

Differentiate (30) and substitute r' from (29) and $\bar{w}'$ from (31) into the result. Assuming that the system is near resonance, i.e., $\bar{w}$ is $O(\varepsilon)$, and neglecting terms of $O(\varepsilon^2)$, we have

$$\chi'' + \frac{\varepsilon r}{2}\sin(\chi) = -\varepsilon K \qquad (32)$$

$$r' = -\frac{\varepsilon}{2}\sin(\chi) \qquad (33)$$

Equation (32) is a pendulum-type equation with slowly varying frequency. When K is a small constant, capture occurs. During capture, χ oscillates about a negative offset between $-90°$ and $0°$, and r grows.

The center of the hole in the resonance plane (Fig. 4) corresponds to χ near 135°. In order to bring χ near 135°, apply a feedback control law which causes (unstable) growing oscillations in χ. This control law is

$$L = -\varepsilon K_{max}\chi' \qquad (34)$$

Recall χ' from (30). Note that $r = \sqrt{\bar{u}^2 + \bar{v}^2} = O(1)$. From the assumption that $\bar{w} = O(\varepsilon)$, we have $\chi' = O(\varepsilon)$, so L defined in Eq. (34) never exceeds the maximum possible torque εK_{max}. When L is substituted from (34) into (32), the result is

$$\chi'' - \varepsilon K_{max}\chi' + \frac{\varepsilon r}{2}\sin(\chi) = 0 \qquad (35)$$

The negative damping in (35) causes growing oscillations in χ. In Fig. 5, this corresponds to spiraling about the w axis. Allow these oscillations to continue

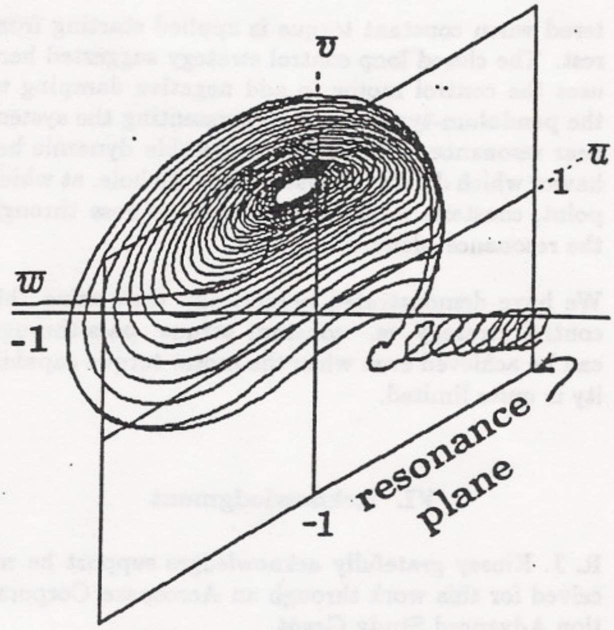

Figure 5: Pass-Through Using Feedback Control

until χ passes within 5° of 135° and r has a magnitude > 0.4, i.e., until the system trajectory is well within the hole in the resonance plane. (See Fig. 4.) At that time, a switch is made from feedback control back to the maximum possible constant torque. Pass-through is achieved after the switch back to constant torque, as seen in Fig. 5.

The general effectiveness of this control strategy is depicted in Figure 3, which contains two curves in the (L, ε) plane. The lower curve (from Eq. (26)) bounds the pass-through region when constant torque ($L = \varepsilon K$) is used. The upper curve bounds the pass-through region when our feedback control strategy is used. It was generated through numerical integration of the exact equations (11) – (14): ε was fixed at various values and the maximum magnitude of L ($= \varepsilon K_{max}$) was varied to determine the value corresponding to marginal pass-through. The curve shown is a least squares fit to 6 points obtained this way, for a range on ε from 0.005 to 0.1.

For a given ε, the value of L for the point (L, ε) on the upper curve is one tenth of the value of L for the point on the lower curve.

V. Summary

There is a region of states near resonance—referred to as the hole in the resonance plane—for which passage through resonance can be achieved even though motor torque is quite limited. This hole is not entered when constant torque is applied starting from rest. The closed loop control strategy suggested here uses the control motor to add negative damping to the pendulum-type equation representing the system near resonance. The result is unstable dynamic behavior which drives the system into the hole, at which point, constant torque can be used to pass through the resonance plane.

We have demonstrated numerically that using this control strategy vs. constant torque, pass-through can be achieved even when the motor torque capability is quite limited.

VI. Acknowledgment

R. J. Kinsey gratefully acknowledges support he received for this work through an Aerospace Corporation Advanced Study Grant.

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