**Proceedings of DETC'99**
**1999 ASME Design Engineering Technical Conferences**
**September 12-15, 1999, Las Vegas, Nevada, USA**

**DETC99/VIB-8052**

# ON THE TORUS FLOW Y' = A + B COS Y + C COS X AND ITS RELATION TO THE QUASIPERIODIC MATHIEU EQUATION

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## ABSTRACT

We obtain power series solutions to the "*abc* equation"

$$\frac{dy}{dx} = a + b\ \cos y + c\ \cos x,$$

valid for small $c$, and for small $b$. This equation is shown to determine the stability of the quasiperiodic Mathieu equation,

$$\ddot{z} + (\delta + \epsilon A_1 \cos t + \epsilon A_2 \cos \omega t)z = 0,$$

in the small $\epsilon$ limit. Perturbation results of the *abc* equation are shown to compare favorably to numerical integration of the quasiperiodic Mathieu equation.

## INTRODUCTION

This work is motivated by an interest in the the quasiperiodic Mathieu equation:

$$\ddot{z} + (\delta + \epsilon A_1 \cos t + \epsilon A_2 \cos \omega t)z = 0, \tag{1}$$

which has been investigated by (Zounes and Rand, 1998) in the case that $A_1 = A_2 = 1$. In particular, we seek expressions for transition surfaces in parameter space which separate regions of stability from regions of instability in Equation (1). (Equation (1) is said to be stable if all solutions remain bounded as $t \to \infty$, and unstable otherwise.)

In this work, we use perturbations valid for small $\epsilon$, small $\omega$, and $\delta \approx 1/4$. Let

$$\delta = \frac{1}{4} + \epsilon\delta_1, \tag{2}$$

$$\omega = \epsilon, \tag{3}$$

$$\tau = \epsilon t. \tag{4}$$

Applying Equations (2)-(4) to Equation (1) gives

$$\ddot{z} + \frac{1}{4}z = \epsilon(-\delta_1 - A_1 \cos t - A_2 \cos \tau)z \tag{5}$$

$$\dot{\tau} = \epsilon. \tag{6}$$

Using the method of averaging to lowest order in $\epsilon$, we seek solutions of the form

$$z = R(t)\cos(\frac{t}{2} + \phi(t)), \tag{7}$$

$$\dot{z} = \frac{-R(t)}{2}\sin(\frac{t}{2} + \phi(t)). \tag{8}$$

Substituting Equations (7) and (8) into Equation (5) gives

$$\dot{R}(t) = -2\epsilon \sin(\frac{t}{2} + \phi(t))F, \tag{9}$$

$$\dot{\phi}(t) = -\frac{2}{R(t)}\epsilon \cos(\frac{t}{2} + \phi(t))F. \tag{10}$$

where $F = -(\delta_1 + A_1 \cos t + A_2 \cos \tau)R\cos(\frac{t}{2} + \phi(t))$.

Averaging Equations (9) and (10) in $t$, while holding $R, \phi$ and $\tau$ constant, gives the slow flow

$$\dot{R} = \frac{A_1}{2}\epsilon R \sin 2\phi \tag{11}$$

$$\dot{\phi} = \epsilon(\frac{A_1}{2}\cos 2\phi + A_2\cos\tau + \delta_1) \tag{12}$$

Solving Equation (11) for $R$ gives

$$R = R_0 \exp\left(\frac{A_1\epsilon}{2}\int_0^t \sin 2\phi \, dt_1\right), \tag{13}$$

where $R_0$ is a constant of integration.

How can we tell the stability of Equation (1) from the averaged slow flow equations? Note that Equations (12) and (6) give an autonomous flow on the $\phi - \tau$ torus. This flow exhibits either (a) limit cycles (which generically come in pairs, one stable and one unstable), or (b) a quasiperiodic flow having no limit cycles. In the limit cycle case (a), $\phi$ will be a periodic function of $t$, and so will $\sin 2\phi$. In such a case, the integral in Equation (13) will in general be non-zero over one cycle, and $R$ will either grow unbounded as $t \to \infty$ or will decay to zero. Since the Wronskian of Equation (1) is constant in time, an exponentially decaying solution must be accompanied by a second linearly independent solution which is exponentially growing. Thus a limit cycle in Equations (12) and (6) means Equation (1) is unstable.

On the other hand, if the torus flow is quasiperiodic, then by ergodicity (Arnold and Avez, 1968) the time mean in Equation (13) is equal to the space mean, that is

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \sin 2\phi \, dt_1 = \frac{1}{\pi\cdot 2\pi}\int_{\phi=0}^{\pi}\int_{\tau=0}^{2\pi}\sin 2\phi \, d\tau d\phi = 0. \tag{14}$$

In this case $R$ will remain bounded as $t \to \infty$ and Equation (1) is stable.

We now choose $\tau$ to be the independent variable in Equations (12) and (6) giving

$$\frac{d\phi}{d\tau} = \frac{d\phi}{dt} \div \frac{d\tau}{dt} = \delta_1 + \frac{A_1}{2}\cos 2\phi + A_2\cos\tau. \tag{15}$$

For convenience of notation, we let

$$\begin{aligned} y &= 2\phi, \\ x &= \tau, \\ a &= 2\delta_1, \\ b &= A_1, \\ c &= 2A_2, \end{aligned} \tag{16}$$

then Equation (15) becomes

$$\frac{dy}{dx} = a + b\ \cos y + c\ \cos x. \tag{17}$$

Table 1. Invariance of Equation (17) under the transformations listed for each octant. Note that none of these operations change the qualitative behavior of Equation (17).

| Octant | Invariance of Equation (17) |
|---|---|
| (Signs of a,b,c) | |
| $1(+,+,+)$ | —— |
| $2(+,+,-)$ | $c \mapsto -c, x \mapsto x-\pi$ |
| $3(+,-,-)$ | $b \mapsto -b, c \mapsto -c, x \mapsto x-\pi,\ y \mapsto y-\pi$ |
| $4(+,-,+)$ | $b \mapsto -b, y \mapsto y-\pi$ |
| $5(-,+,+)$ | $a \mapsto -a, x \mapsto x-\pi,\ y \mapsto \pi - y$ |
| $6(-,+,-)$ | $a \mapsto -a, c \mapsto -c, y \mapsto \pi - y$ |
| $7(-,-,-)$ | $a \mapsto -a, b \mapsto -b, c \mapsto -c, y \mapsto -y$ |
| $8(-,-,+)$ | $a \mapsto -a, b \mapsto -b, x \mapsto x-\pi,\ y \mapsto -y$ |

We shall refer to Equation (17), which gives a slope field on the $x - y$ torus, as the "abc equation". In summary, if the solution to Equation (17) on the torus possesses a limit cycle, then the quasiperiodic Mathieu equation (1) is unstable; if the abc equation is quasiperiodic (no limit cycles), then Equation (1) is stable. Note that these results are based on the method of averaging, and assume that $\epsilon << 1$.

## THE ABC EQUATION

In this work we shall be interested in the question of which points in $a - b - c$ parameter space correspond to stable solutions (no limit cycles in the abc equation), and which correspond to unstable solutions (limit cycles). We begin by noting that it is sufficient to study the first octant ($a \geq 0$, $b \geq 0$, and $c \geq 0$) only. This is because Equation (17) is invariant under a variety of translations of $x$ and $y$ and sign changes in $y$. For example, in the case of the second octant, Equation (17) is invariant under the transformations $c \mapsto -c$ and $x \mapsto x - \pi$. See Table 1. In the remainder of this work we shall use perturbation theory to study the abc equation for small values of $c$ and $b$.

### Small c Approximation

When $c = 0$, Equation (17) becomes

$$\frac{dy}{dx} = a + b\ \cos y. \tag{18}$$

Note that Equation (18) is autonomous, that is, independent of $x$. When Equation (18) is viewed as a flow on the $y$-circle, the $y$ values for which $dy/dx = 0$ are fixed points. But if we view Equation (18) as a flow on the $x$-$y$ torus, then $y$ values for which $dy/dx = 0$ are limit cycles. Therefore a limit cycle for Equation (18) corresponds to the $y$ values for which

$$a + b\ \cos y = 0 \Rightarrow \cos y = -\frac{a}{b}. \tag{19}$$

For real positive solutions to Equation (19) we require

$$b \geq a. \tag{20}$$

Thus for small values of $c$, parameters satisfying Equation (20) are unstable.

If $b < a$, then for $c = 0$ all solutions to the abc equation are in general quasiperiodic and such points in parameter space are stable. However, KAM theory (Guckenheimer and Holmes, 1983) tells us that for small values of $c$, tongues of instability will emanate from points in the $a - b$ plane for which the winding number of the (non-limit cycle) orbit is rational. In order to calculate the winding number, we write Equation (18) in the form:

$$\frac{dy}{dt} = a + b\ \cos y, \qquad \frac{dx}{dt} = 1, \tag{21}$$

which has the exact solution:

$$x(t) = t,\ y(t) = \arctan\left(\frac{2\sigma \tan\frac{\sigma t}{2}}{(a+b)\tan^2\frac{\sigma t}{2} + (b-a)}\right) \tag{22}$$

where $\sigma = \sqrt{a^2 - b^2}$, and where the constants of integration have been chosen to satisfy the initial condition $x(0) = 0, y(0) = 0$. Now we consider the Poincaré map with surface of section $x = 0 \mod 2\pi$. Setting $t = 2n\pi$, where $n$ is an integer, we obtain the circle map:

$$y_n = \arctan\left(\frac{2\sigma \tan\sigma n\pi}{(a+b)\tan^2\sigma n\pi + (b-a)}\right) \tag{23}$$

The winding number $\rho$ is given by

$$\rho = \lim_{n\to\infty}\frac{y_n}{2\pi n} = \sigma = \sqrt{a^2 - b^2} \tag{24}$$

where we have used the fact that $y_n$ changes by $2\pi$ when the argument of $\tan\sigma n\pi$ changes by $\pi$.

From Equation (24) and KAM theory we conclude that as $c$ is increased from zero, tongues of instability emanate from the following curves in the $a - b$ plane:

$$\rho = \sqrt{a^2 - b^2} = \frac{m}{n} \qquad \Rightarrow \qquad a = \sqrt{\frac{m^2}{n^2} + b^2} \tag{25}$$

where $m$ and $n$ are relatively prime positive integers. See Figure 1. This result illustrates an essential difference between the quasiperiodic Mathieu equation (1) and the usual Mathieu equation (i.e. Equation (1) with $A_2 = 0$), in the small $\epsilon$ limit. Namely that the former has a dense set of instability regions, whereas although the latter has an infinitude of instability regions, these are spaced a finite distance away from one another (Stoker, 1950).

Now we will use perturbations to obtain a small-c approximation for the transition surface which bounds the region containing limit cycles. For $c = 0$ this is given by the equation

$$a = b, \qquad c = 0. \tag{26}$$

This transition is characterized by a saddle-node bifurcation and the occurrence of a degenerate (semistable) limit cycle. In the $c = 0$ case this occurs when

$$\cos y = -\frac{a}{b} = -1 \qquad \Rightarrow \qquad y = \pi. \tag{27}$$

When $c > 0$, this transition continues to be characerized by the occurrence of a degenerate limit cycle. Using symmetry properties of the abc equation, we prove in the Appendix that this limit cycle satisfies the condition

$$y(0) = \pi. \tag{28}$$

Generalizing Equation (26) for $c > 0$, we expand $a$ around $b$ in a power series in $c$. Substituting the equations

$$a = b + ck_1 + c^2 k_2 + \ldots \tag{29}$$

$$y = y_0 + cy_1 + c^2 y_2 + \ldots \tag{30}$$

into Equation (17), expanding for small $c$ and collecting terms, we obtain a sequence of differential equations, the first three of which are:

$$c^0\colon\ \frac{dy_0}{dx} = b(1 + \cos y_0) \tag{31}$$

$$c^1\colon\ \frac{dy_1}{dx} = k_1 + \cos x - by_1 \sin y_0 \tag{32}$$

$$c^2\colon\ \frac{dy_2}{dx} = \frac{2k_2 - ({y_1}^2 \cos y_0 + 2y_2 \sin y_0)b}{2}. \tag{33}$$

From Equation (27) we see that the appropriate solution to Equation (31) is $y_0(x) = \pi$. Substituting this into Equation (32) we obtain:

$$y_1(x) = k_1 x + \sin x + \gamma_1 \tag{34}$$

where $\gamma_1$ is a constant of integration. In order for the perturbation solution to be uniformly valid as $x \to \infty$, we must remove any secular terms which grow linearly in $x$. Thus we take $k_1 = 0$. Also, Equation (28) gives $\gamma_1 = 0$. Thus we find that $y_1(x) = \sin x$. Substituting $y_0(x)$ and $y_1(x)$ into Equation (33) we obtain:

$$\frac{dy_2}{dx} = \frac{b}{2}\sin^2 x + k_2. \tag{35}$$

Using the identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ in Equation (35), we require $k_2 = -\frac{b}{4}$ for no secular terms. Thus Equation (29) gives the following expression for the transition surface:

$$a = b - \frac{b}{4}c^2 + O(c^3). \tag{36}$$

This process can be extended to obtain a higher order expression for the transition surface. For example,

$$a = b - c^2\frac{b}{4} + c^4(\frac{4b + 7b^3}{256}) - c^6(\frac{4b + 73b^3 + 58b^5}{9216}) + O(c^8). \tag{37}$$

**Small b Approximation**

When $b = 0$, Equation (17) becomes

$$\frac{dy}{dx} = a + c\ \cos x, \tag{38}$$

which has the solution:

$$y(x) = ax + c\ \sin x + \mu, \tag{39}$$

where $\mu$ is an integration constant. Since the phase space is a torus, the unperturbed solution (39) will represent a periodic motion with winding number $m/n$ when

$$a = \frac{m}{n}, \tag{40}$$

where $m$ and $n$ are relatively prime positive integers. To see what happens to this periodic solution for small values of $b$, we expand $a$ and $y$ in power series in $b$ (for a fixed value of $c$):

$$a = a_0 + a_1 b + \ldots, \text{ where } a_0 = \frac{m}{n} \tag{41}$$

$$y = y_0 + y_1 b + \ldots, \text{ where } y_0(x) = a_0 x + c\sin x + \mu \tag{42}$$

Substituting (41) and (42) into Equation (17), expanding and collecting terms yields:

$$\frac{dy_1}{dx} = a_1 + \cos y_0 \tag{43}$$

Equation (43) may be integrated to give

$$y_1(x) = y_1(0) + a_1 x + \int_0^x \cos(a_0 u + c\ \sin u + \mu)\ du \tag{44}$$

which may be written in the form

$$y_1(x) = y_1(0) + a_1 x + \cos\mu \int_0^x \cos(a_0 u + c\ \sin u)\ du - \sin\mu \int_0^x \sin(a_0 u + c\ \sin u)\ du, \tag{45}$$

where $y_1(0)$ is an integration constant. Now we require the perturbed motion to continue to be periodic with winding number $\frac{m}{n}$. Such a motion will correspond to an $n-$cycle in the Poincaré map with surface of section $\sum : x = 0 \bmod 2\pi$. Thus we require

$$y(0) = y(2n\pi) \qquad \text{mod} 2\pi \tag{46}$$

$$y_0(0) + y_1(0)b + \cdots = y_0(2n\pi) + y_1(2n\pi)b + \cdots \quad \text{mod} 2\pi, \tag{47}$$

$$y_0(0) = y_0(2n\pi), \qquad y_1(0) = y_1(2n\pi) \quad \text{mod} 2\pi \tag{48}$$

Note that from Equations (42) and (45)

$$y_0(2n\pi) = 2n\pi a_0 + \mu = 2\pi m + \mu, \qquad y_0(0) = \mu, \tag{49}$$

$$y_1(2n\pi) = y_1(0) + 2n\pi a_1 + \cos\mu \int_0^{2n\pi} \cos(\frac{m}{n}u + c\ \sin u)\ du - \sin\mu \int_0^{2n\pi} \sin(\frac{m}{n}u + c\ \sin u)\ du. \qquad (50)$$

Now Equations (49) identically satisfy the first of Equations (48). Also, in Equation (50), $\int_0^{2n\pi} \sin(\frac{m}{n}u + c\ \sin u)\ du = 0$ since the integrand is an odd function. Thus the second of Equations (48) requires

$$a_1 = -\frac{1}{2n\pi}\cos\mu \int_0^{2n\pi} \cos(\frac{m}{n}u + c\ \sin u)\ du \qquad (51)$$

The transition surface separating points in parameter space which have limit cycles from those which don't corresponds to the limiting values of $\cos\mu$, namely $\cos\mu = \pm 1$, which gives:

$$a_1 = \mp\frac{1}{2n\pi} \int_0^{2n\pi} \cos(\frac{m}{n}u + c\ \sin u)\ du \qquad (52)$$

In the special case of $n = 1$, the integral in Equation (52) reduces to a Bessel function of the first kind (Abramowitz and Stegun, 1965):

$$a_1 = \mp J_m(-c), \qquad n = 1. \qquad (53)$$

The expression for the transition surfaces becomes:

$$a = \frac{m}{n} \mp b\frac{1}{2n\pi} \int_0^{2n\pi} \cos(\frac{m}{n}u + c\ \sin u)\ du + O(b^2) \qquad (54)$$

We have extended these results to include $O(b^2)$ terms, but we omit the associated expressions here to save space. For $n = 1,\ m = 1,\ c = 1$, we obtained

$$a = 1 \pm 0.44005\ b + 0.3367\ b^2 + O(b^3) \qquad (55)$$

Figure 2 shows the perturbation results (37) and (55).

## NUMERICAL INTEGRATION

In order to check the foregoing perturbation results we numerically integrated the quasiperiodic Mathieu equation (1) in conjunction with Floquet theory (Stoker, 1950) for a fine mesh of rational values of $\delta$ and $A_1$. The results for $\omega = \epsilon = 0.1$ and $A_2 = 0.5$ are displayed in Figure 3, which should be compared to the perturbation results in Figure 2. We note that the numerous instability regions predicted by KAM theory in Figure 1 do not show up in Figures 2 and 3. This is due to their being very thin for $\epsilon = 0.1$.

## CONCLUSIONS

In this work we have used the method of averaging to investigate the dynamics of the quasiperiodic Mathieu equation (1). Using the ergodic theorem, we showed that a necessary and sufficient condition for Equation (1) to be stable is that the resulting slow flow, called the *abc* equation (17), must not possess a limit cycle. We used KAM theory to show that the *abc* equation (17) exhibited a dense set of points in the *ab* parameter plane from which instability arises via Arnold tongues. See Figure 1. This result supplements previous work (Zounes and Rand, 1998) in which it was shown that as the order of the perturbation method increased, the number and density of instability regions increased.

Comparison of the analytical results obtained from the *abc* equation (17) with numerical integration of the quasiperiodic Mathieu equation (1) shows good agreement for $\epsilon = 0.1$, cf. Figures 2 and 3. These figures also show that for small $\epsilon$ most of the instability regions (for $n > 1$) are very small, to the point of being too thin to be seen in Figures 2 and 3. In addition, it was found that the points on the $a-$axis from which the instability tongues emerge, predicted by perturbation theory to occur at $a = \frac{m}{n}$, actually occur (for finite values of $\epsilon$) at larger values of $a$, cf. Figures 2 and 3.


## ACKNOWLEDGEMENT

The authors wish to thank Professors John Guckenheimer and Steven Strogatz for helpful discussions. Thanks also to Stephen Wirkus for reading the manuscript.


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## APPENDIX: DEGENERATE LIMIT CYCLE

**Lemma 1.** *If $y = f(x)$ is a solution to Equation (17), then $y = -f(-x)$ is also a solution to Equation (17).*

**Proof.** Let $y = f(x)$ be a solution to Equation (17):

$$\frac{df(x)}{dx} = a + b \ \cos f(x) + c \ \cos x, \tag{56}$$

where $a$, $b$, and $c$ are real-valued parameters. Replacing $x$ by $-x$ in Equation (56) gives

$$\frac{df(-x)}{d(-x)} = a + b \ \cos f(-x) + c \ \cos(-x). \tag{57}$$

Using the chain rule on the left and noting that cosine is an even function gives

$$\frac{d(-f(-x))}{dx} = a + b \ \cos(-f(-x)) + c \ \cos x. \tag{58}$$

Writing $g(x) = -f(-x)$, Equation (58) becomes

$$\frac{dg(x)}{dx} = a + b \ \cos g(x) + c \ \cos x. \tag{59}$$

Equations (17) and (59) have the same form, therefore $y = g(x) = -f(-x)$ is a solution to Equation (17).

**Theorem 1.** *The degenerate limit cycle of Equation (17) has the property that $y(0) = \pi$.*

**Proof.** When $c = 0$ and $0 \leq a < b$ Equation (17) has two limit cycles; see Equation (19). For small $c$, by structural stability, we are guaranteed that two limit cycles still exist but now for nearby values of $a$ and $b$.

Consider parameter values $a$, $b$, and $c$ such that Equation (17) has two limit cycles. Let the initial condition for one of the limit cycles, $y = f(x)$, be

$$y(0) = f(0) = A. \tag{60}$$

From Lemma (1), since the limit cycle $y = f(x)$ is a solution, $y = -f(-x)$ is also a solution. Since $y = f(x)$ is a periodic solution (a limit cycle), $y = -f(-x)$ must also be a limit cycle. Therefore, the second limit cycle has the initial condition

$$y(0) = -f(0) = -A. \tag{61}$$

But the vector field is periodic with period $2\pi$, therefore Equation (61) can be written as

$$y(0) = -A = (2\pi - A)\mathrm{mod}2\pi. \tag{62}$$

Now what happens as the parameters $a$, $b$, and $c$ approach a value at which the two limit cycles coalesce into a degenerate limit cycle? Since the two limit cycles merge, their initial conditions must also merge. Therefore from Equations (60) and (62),

$$y(0) = A = 2\pi - A \quad \Rightarrow \quad A = \pi \quad \Rightarrow \quad y(0) = \pi \tag{63}$$

for the degenerate limit cycle.

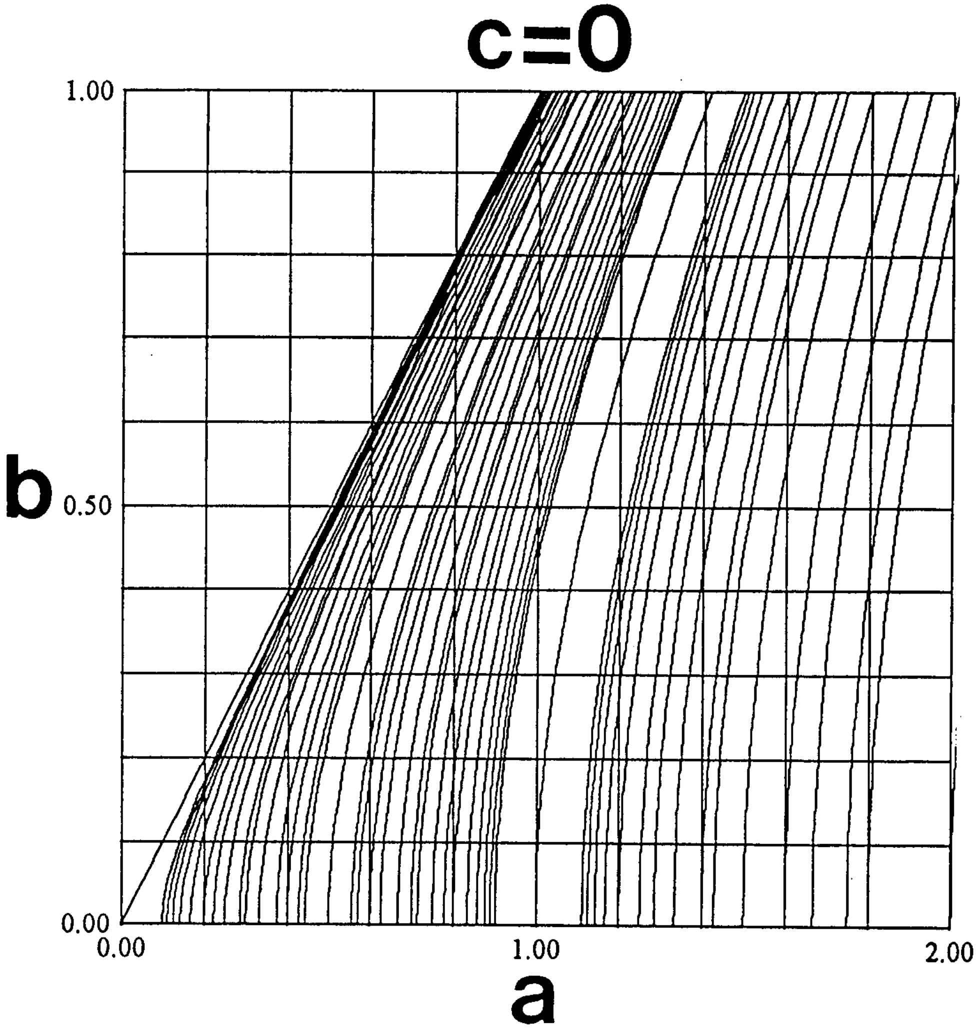


Figure 1. The curves $a = \sqrt{\frac{m^2}{n^2} + b^2}$, where $m$ and $n$ are relatively prime positive integers, Equation (25), plotted in the $a-b$ plane for $c = 0$. For clarity of presentation, only those curves which correspond to values of $n \leq 10$ are displayed. If we imagine the $c$-axis coming out of the page, then KAM theory shows that as $c$ is increased from zero, tongues of instability emanate from each of these curves. Comparison with Figures 2 and 3 (which correspond to $c = 1$) shows that for $c = 1$ all of these tongues are extremely small, except for those that emanate from curves corresponding to $n = 1$, i.e. $a = \sqrt{1 + b^2}$.

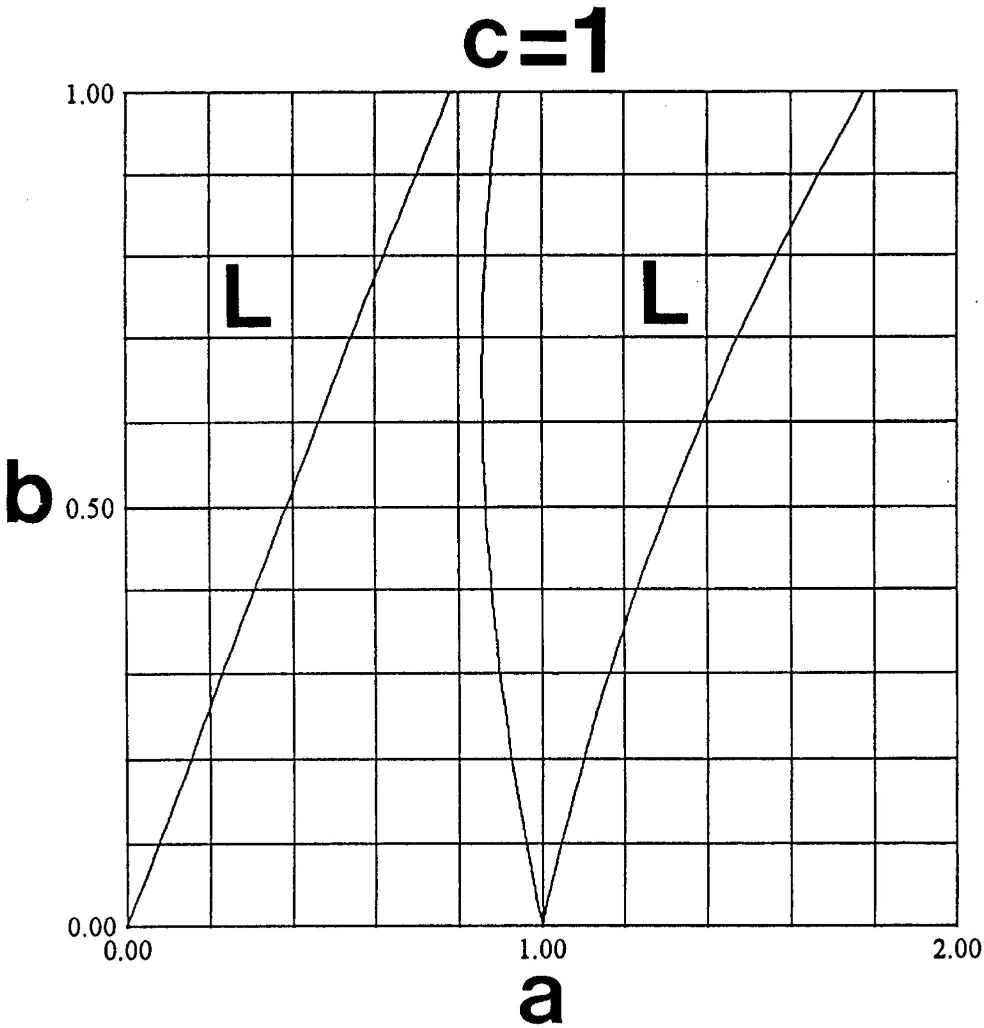


Figure 2. Results of perturbation theory applied to the *abc* equation (17), namely Equations (37) and (55) displayed for $c = 1$. The regions marked $L$ correspond to *abc* equation torus flows (17) which include a limit cycle. Perturbation theory shows that such points correspond to instability in the quasiperiodic Mathieu equation (1), cf. Figure 3.

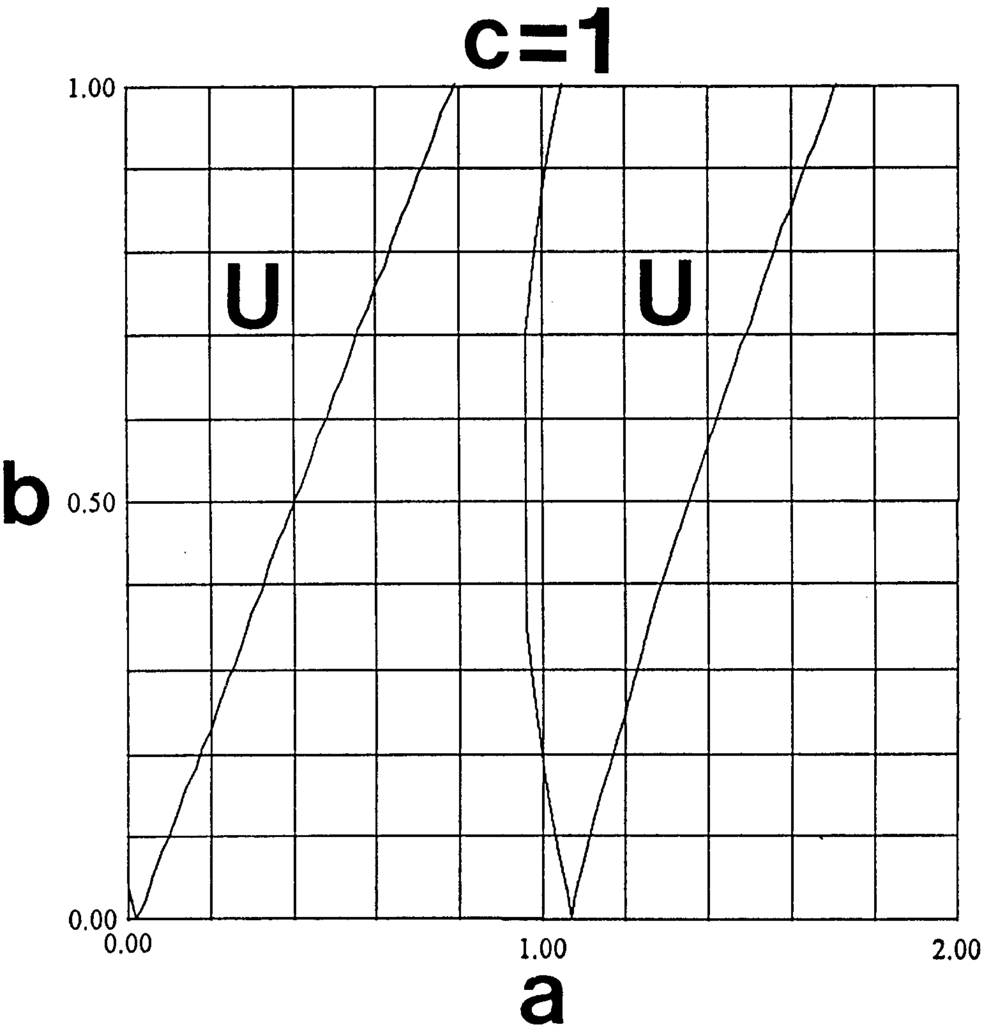


Figure 3. Results of numerical integration of the quasiperiodic Mathieu equation (1) for $\omega = \epsilon = 0.1$, $A_2 = 0.5$. The values of $\delta$ and $A_1$ are varied, and are displayed in terms of the parameters $a, b, c$ by using Equations (2),(16), namely $\delta = 0.25 + 0.05\ a$, $A_1 = b$, $c = 2A_2 = 1$. The regions marked $U$ correspond to instability (unbounded solutions) in the quasiperiodic Mathieu equation (1).