Mathieu's Equation

Richard H. Rand
Cornell University

The differential equation

$$\frac{d^2x}{dt^2} + (\delta + \epsilon \cos t)\ x = 0 \tag{1}$$

is called Mathieu's equation. It is a linear differential equation with variable (periodic) coefficients. It commonly occurs in nonlinear vibration problems in two different ways: (i) in systems in which there is periodic forcing, and (ii) in stability studies of periodic motions in nonlinear *autonomous* systems.

As an example of (i), take the case of a pendulum whose support is periodically forced in a vertical direction. The governing differential equation is

$$\frac{d^2x}{dt^2} + \left(\frac{g}{L} - \frac{A\omega^2}{L}\cos\omega t\right)\sin x = 0 \tag{2}$$

where the vertical motion of the support is $A\cos\omega t$, and where g is the acceleration of gravity, L is the pendulum's length, and x is its angle of deflection. In order to investigate the stability of one of the equilibrium solutions $x = 0$ or $x = \pi$, we would linearize (2) about the desired equilibrium, giving, after suitable rescaling of time, an equation of the form of (1).

As an example of (ii), we consider a system known as "the particle in the plane". This consists of a particle of unit mass which is constrained to move in the x-y plane, and is restrained by two linear springs, each with spring constant of $\frac{1}{2}$. The anchor points of the two springs are located on the x axis at $x = 1$ and $x = -1$. Each of the two springs has unstretched length L. This autonomous two degree of freedom system exhibits an exact solution corresponding to a mode of vibration in which the particle moves along the x axis:

$$x = A\cos t, \qquad y = 0 \tag{3}$$

In order to determine the stability of this motion, one must first derive the equations of motion, then substitute $x = A\cos t + u, \quad y = 0 + v$, where u and v are small deviations from the motion (3), and then linearize in u and v. The result is two linear differential equations on u and v. The u equation turns out to be the simple harmonic oscillator, and cannot produce instability. The v equation is:

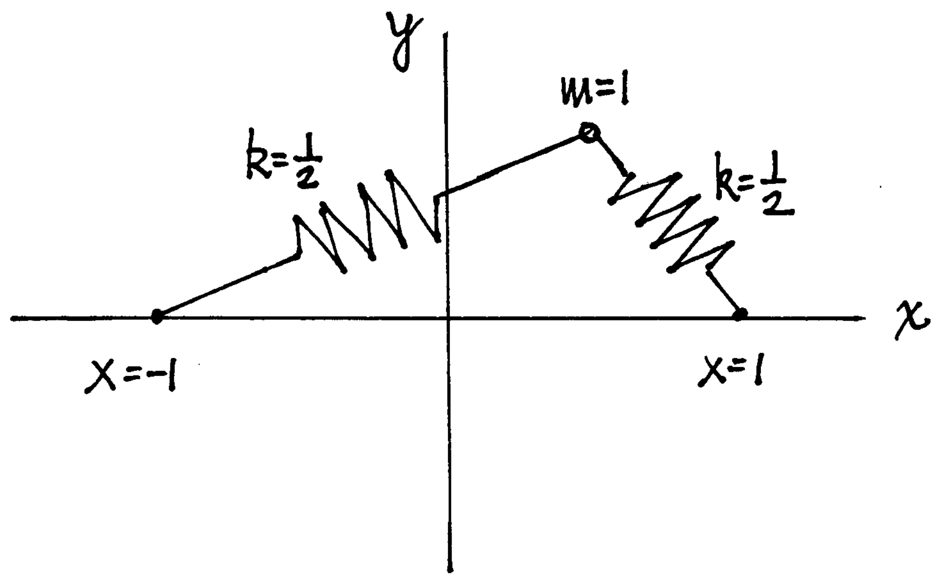
$$\frac{d^2v}{dt^2} + \left(\frac{1 - L - A^2\cos^2 t}{1 - A^2\cos^2 t}\right) v = 0 \tag{4}$$

Expanding (4) for small A and setting $\tau = 2t$, we obtain

$$\frac{d^2v}{d\tau^2} + \left(\frac{2 - 2L - A^2L}{8} - \frac{A^2L}{8}\cos\tau + O(A^4)\right) v = 0 \tag{5}$$

which is, to $O(A^4)$, in the form of Mathieu's eq.(1) with $\delta = \dfrac{2 - 2L - A^2L}{8}$ and $\epsilon = -\dfrac{A^2L}{8}$.

The particle in the plane



$$\frac{d^2x}{dt^2} + f_1(x,y)\ (x+1) + f_2(x,y)\ (x-1) = 0$$

$$\frac{d^2y}{dt^2} + f_1(x,y)\ y + f_2(x,y)\ y = 0$$

$$f_1(x,y) = \frac{1}{2}\left(1 - \frac{L}{\sqrt{(1+x)^2+y^2}}\right)$$

$$f_2(x,y) = \frac{1}{2}\left(1 - \frac{L}{\sqrt{(1-x)^2+y^2}}\right)$$

The chief concern with regard to Mathieu's equation is whether or not all solutions are bounded for given values of the parameters δ and ϵ. If all solutions are bounded then the corresponding point in the δ-ϵ parameter plane is said to be stable. A point is called unstable if an unbounded solution exists.

Perturbations

In this section we will use the two variable expansion method to look for a general solution to Mathieu's eq.(1) for small ϵ. Since (1) is linear, there is no need to stretch time, and we set $\xi = t$ and $\eta = \epsilon t$, giving

$$\frac{\partial^2 x}{\partial \xi^2} + 2\epsilon \frac{\partial^2 x}{\partial \xi \partial \eta} + \epsilon^2 \frac{\partial^2 x}{\partial \eta^2} + (\delta + \epsilon \cos \xi)\ x = 0 \tag{6}$$

Next we expand x in a power series:

$$x(\xi, \eta) = x_0(\xi, \eta) + \epsilon x_1(\xi, \eta) + \cdots \tag{7}$$

Substituting (7) into (1) and neglecting terms of $O(\epsilon^2)$, gives, after collecting terms:

$$\frac{\partial^2 x_0}{\partial \xi^2} + \delta\ x_0 = 0 \tag{8}$$

$$\frac{\partial^2 x_1}{\partial \xi^2} + \delta\ x_1 = -2\frac{\partial^2 x_0}{\partial \xi \partial \eta} - x_0 \cos \xi \tag{9}$$

We take the general solution to eq.(8) in the form:

$$x_0(\xi, \eta) = A(\eta) \cos \sqrt{\delta}\ \xi + B(\eta) \sin \sqrt{\delta}\ \xi \tag{10}$$

Substituting (10) into (9), we obtain

$$\begin{aligned} \frac{\partial^2 x_1}{\partial \xi^2} + \delta\ x_1 &= 2\sqrt{\delta}\frac{dA}{d\eta} \sin \sqrt{\delta}\ \xi - 2\sqrt{\delta}\frac{dB}{d\eta} \cos \sqrt{\delta}\ \xi \\ &\quad -A \cos \sqrt{\delta}\ \xi \cos \xi - B \sin \sqrt{\delta}\ \xi \cos \xi \end{aligned} \tag{11}$$

Using some trig identities, this becomes

$$\begin{aligned} \frac{\partial^2 x_1}{\partial \xi^2} + \delta\ x_1 &= 2\sqrt{\delta}\frac{dA}{d\eta} \sin \sqrt{\delta}\ \xi - 2\sqrt{\delta}\frac{dB}{d\eta} \cos \sqrt{\delta}\ \xi \\ &\quad -\frac{A}{2}\left(\cos(\sqrt{\delta}\ +1)\xi + \cos(\sqrt{\delta}\ -1)\xi\right) \\ &\quad -\frac{B}{2}\left(\sin(\sqrt{\delta}\ +1)\xi + \sin(\sqrt{\delta}\ -1)\xi\right) \end{aligned} \tag{12}$$

For a general value of δ, removal of resonance terms gives the trivial slow flow:

$$\frac{dA}{d\eta} = 0, \qquad \frac{dB}{d\eta} = 0 \tag{13}$$

This means that for general δ, the $\cos t$ driving term in Mathieu's eq.(1) has no effect. However, if we choose $\delta = \frac{1}{4}$, eq.(12) becomes

$$\begin{aligned}\frac{\partial^2 x_1}{\partial \xi^2} + \frac{1}{4}\, x_1 \;&=\; \frac{dA}{d\eta} \sin\frac{\xi}{2} - \frac{dB}{d\eta}\cos\frac{\xi}{2} \\ &\quad -\frac{A}{2}\left(\cos\frac{3\xi}{2} + \cos\frac{\xi}{2}\right) \\ &\quad -\frac{B}{2}\left(\sin\frac{3\xi}{2} - \sin\frac{\xi}{2}\right) \qquad (14)\end{aligned}$$

Now removal of resonance terms gives the slow flow:

$$\frac{dA}{d\eta} = -\frac{B}{2}, \qquad \frac{dB}{d\eta} = -\frac{A}{2} \qquad \Rightarrow \qquad \frac{d^2A}{d\eta^2} = \frac{A}{4} \qquad (15)$$

Thus $A(\eta)$ and $B(\eta)$ involve exponential growth, and the parameter value $\delta = \frac{1}{4}$ causes instability. This corresponds to a 2:1 subharmonic resonance in which the driving frequency is twice the natural frequency.

This discussion may be generalized by "detuning" the resonance, that is, by expanding δ in a power series in ϵ:

$$\delta = \frac{1}{4} + \delta_1\epsilon + \delta_2\epsilon^2 + \cdots \qquad (16)$$

Now eq.(9) gets an additional term:

$$\frac{\partial^2 x_1}{\partial \xi^2} + \frac{1}{4}\, x_1 = -2\frac{\partial^2 x_0}{\partial\xi\partial\eta} - x_0\cos\xi - \delta_1 x_0 \qquad (17)$$

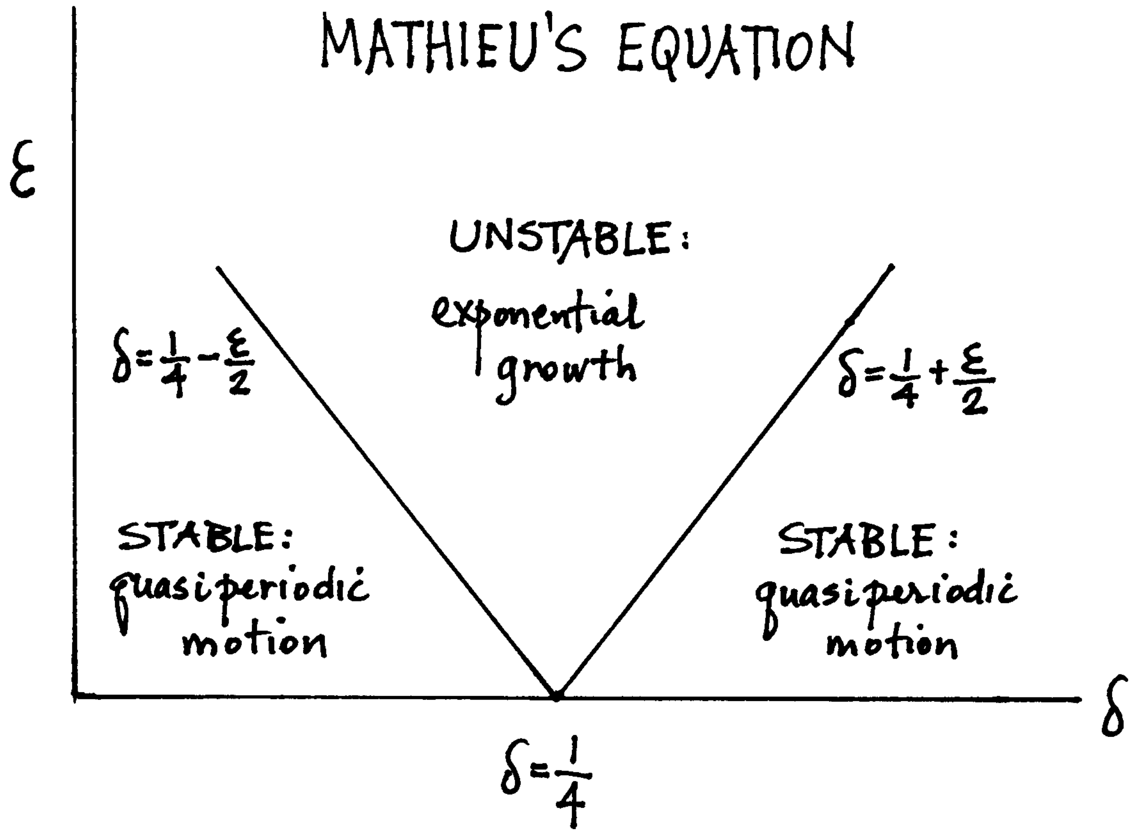
which results in the following additional terms in the slow flow eqs.(15):

$$\frac{dA}{d\eta} = \left(\delta_1 - \frac{1}{2}\right)B, \qquad \frac{dB}{d\eta} = -\left(\delta_1 + \frac{1}{2}\right)A \qquad \Rightarrow \qquad \frac{d^2A}{d\eta^2} + \left(\delta_1^2 - \frac{1}{4}\right)\, A = 0 \qquad (18)$$

Here we see that $A(\eta)$ and $B(\eta)$ will be sine and cosine functions of slow time η if $\delta_1^2 - \dfrac{1}{4} > 0$, that is, if either $\delta_1 > \dfrac{1}{2}$ or $\delta_1 < -\dfrac{1}{2}$. Thus the following two curves in the δ-ϵ plane represent stability changes, and are called *transition curves:*

$$\delta = \frac{1}{4} \pm \frac{\epsilon}{2} + O(\epsilon^2) \qquad (19)$$

These two curves emanate from the point $\delta = \frac{1}{4}$ on the δ axis and define a region of instability called a *tongue.* Inside the tongue, for small ϵ, x grows exponentially in time. Outside the tongue, from (10) and (18), x is the sum of terms each of which is the product of two periodic (sinusoidal) functions with generally incommensurate frequencies, that is, x is a quasiperiodic function of t.



MATHIEU'S EQUATION
ε
UNSTABLE:
exponential growth
$\delta = \frac{1}{4} - \frac{\varepsilon}{2}$
$\delta = \frac{1}{4} + \frac{\varepsilon}{2}$
STABLE:
quasiperiodic motion
STABLE:
quasiperiodic motion
δ
$\delta = \frac{1}{4}$

Floquet Theory

In this section we present Floquet theory, that is, the general theory of linear differential equations with periodic coefficients. Our goal is to apply this theory to Mathieu's equation (1).

Let x be an $n \times 1$ column vector, and let A be an $n \times n$ matrix with time-varying coefficients which have period T. Floquet theory is concerned with the following system of first order differential equations:

$$\frac{dx}{dt} = A(t)\ x, \qquad A(t+T) = A(t) \tag{20}$$

Notice that if the independent variable t is replaced by $t+T$, the system (20) remains invariant. This means that if $x(t)$ is a solution (vector) of (20), and if in the vector function $x(t)$, t is replaced everywhere by $t+T$, then new vector, $x(t+T)$, which in general will be completely different from $x(t)$, is also a solution of (20). This observation may be stated conveniently in terms of fundamental solution matrices.

Let $X(t)$ be a fundamental solution matrix of (20). $X(t)$ is then an $n \times n$ matrix, with each of its columns consisting of a linearly independent solution vector of (20). In particular, we choose the i^{th} column vector to satisfy an initial condition for which each of the scalar components of $x(0)$ is zero, except for the i^{th} scalar component of $x(0)$, which is unity. This gives $X(0) = I$, where I is the $n \times n$ identity matrix. Since the columns of $X(t)$ are linearly independent, they form a basis for the n-dimensional solution space of (20), and thus any other fundamental solution matrix $Z(t)$ may be written in the form $Z(t) = X(t)\ C$, where C is a nonsingular $n \times n$ matrix. This means that each of the columns of $Z(t)$ may be written as a linear combination of the columns of $X(t)$.

From our previous observations, replacing t by $t+T$ in $X(t)$ produces a new fundamental solution matrix $X(t+T)$. Each of the columns of $X(t+T)$ may be written as a linear combination of the columns of $X(t)$, so that

$$X(t+T) = X(t)\ C \tag{21}$$

Note that at $t = 0$, (21) becomes $X(T) = X(0)C = IC = C$, that is,

$$C = X(T) \tag{22}$$

Eq.(22) says that the matrix C (about which we know nothing up to now) is in fact equal to the value of the fundamental solution matrix $X(t)$ evaluated at time T, that is, after one forcing period. Thus C could be obtained by numerically integrating (20) from $t = 0$ to $t = T$, n times, once for each of the n initial conditions satisfied by the i^{th} column of $X(0)$.

Eq.(21) is a key equation here. It has replaced the original system of o.d.e.'s with an iterative equation. For example, if we were to consider eq.(21) for the set of t values $t = 0, T, 2T, 3T, \cdots$, we would be generating the successive iterates of a Poincare map corresponding to the surface of section $\Sigma : t = 0 \ (\mathrm{mod} 2\pi)$. This immediately gives the result that $X(nT) = C^n$, which shows that the question of the boundedness of solutions is intimately connected to the matrix C.

In order to solve eq.(21), we transform to normal coordinates. Let $Y(t)$ be another fundamental solution matrix, as yet unknown. Each of the columns of $Y(t)$ may be written as a linear combination of the columns of $X(t)$:

$$Y(t) = X(t)\ R \tag{23}$$

where R is an as yet unknown $n \times n$ nonsingular matrix. Combining eqs.(21) and (23), we obtain

$$Y(t+T) = Y(t)\ R^{-1}CR \tag{24}$$

Now let us suppose that the matrix C has n linearly independent eigenvectors. If we choose the columns of R as these n eigenvectors, then the matrix product $R^{-1}CR$ will be a diagonal matrix with the eigenvalues λ_i of C on its main diagonal. With $R^{-1}CR$ diagonal, the matrix $Y(t)$ satisfying (24) will also be diagonal. This can be shown by construction: Let $y_i(t)$ represent the i^{th} scalar component on the main diagonal of $Y(t)$. Then assuming $Y(t)$ is diagonal, (24) can be written:

$$y_i(t+T) = \lambda_i\ y_i(t) \tag{25}$$

Eq.(25) is a linear functional equation. Let us look for a solution to it in the form

$$y_i(t) = \lambda_i^{kt}\ p_i(t) \tag{26}$$

where k is an unknown constant and $p_i(t)$ is an unknown function. Substituting (26) into (25) gives:

$$y_i(t+T) = \lambda_i^{k(t+T)}\ p_i(t+T) = \lambda_i(\lambda_i^{kt}\ p_i(t)) = \lambda_i\ y_i(t) \tag{27}$$

Eq.(27) is satisfied if we take $k = 1/T$ and $p_i(t)$ a periodic function of period T:

$$y_i(t) = \lambda_i^{t/T}\ p_i(t), \qquad p_i(t+T) = p_i(t) \tag{28}$$

Here eq.(28) is the general solution to eq.(25). The arbitrary periodic function $p_i(t)$ plays the same role here that an arbitrary constant plays in the case of a linear first order o.d.e.

Since we are interested in the question of boundedness of solutions, we can see from eq.(28) that if $|\lambda_i| > 1$, then $y_i \to \infty$ as $t \to \infty$, whereas if $|\lambda_i| < 1$, then $y_i \to 0$ as $t \to \infty$. Thus we see that the original system (20) will be stable (all solutions bounded) if every eigenvalue λ_i of $C = X(T)$ has modulus less than unity. If any one eigenvalue λ_i has modulus greater than unity, then (20) will be unstable (an unbounded solution exists).

Note that our assumption that C has n linearly independent eigenvectors could be relaxed, in which case we would have to deal with Jordan canonical form. The reader is referred to "Asymptotic Behavior and Stability Problems in Ordinary Differential Equations" by L.Cesari, Springer Verlag, 1963, section 4.1 for a complete discussion of this case.

Hill's Equation

In this section we apply Floquet theory to a generalization of Mathieu's equation (1), called Hill's equation:

$$\frac{d^2x}{dt^2} + f(t)\ x = 0, \qquad f(t+T) = f(t) \tag{29}$$

Here x and f are scalars, and $f(t)$ represents a general periodic function with period T. Eq.(29) includes examples such as eq.(4).

We begin by defining $x_1 = x$ and $x_2 = \dfrac{dx}{dt}$ so that (29) can be written as a system of two first order o.d.e.'s:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -f(t) & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{30}$$

Next we construct a fundamental solution matrix out of two solution vectors, $\begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix}$ and $\begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix}$, which satisfy the initial conditions:

$$\begin{bmatrix} x_{11}(0) \\ x_{12}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_{21}(0) \\ x_{22}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{31}$$

As we saw in the previous section, the matrix C is the evaluation of the fundamental solution matrix at time T:

$$C = \begin{bmatrix} x_{11}(T) & x_{21}(T) \\ x_{12}(T) & x_{22}(T) \end{bmatrix} \tag{32}$$

From Floquet theory we know that stability is determined by the eigenvalues of C:

$$\lambda^2 - (\mathrm{tr}C)\lambda + \det C = 0 \tag{33}$$

where tr$C$ and det$C$ are the trace and determinant of C. Now Hill's eq.(29) has the special property that det$C$=1. This may be shown by defining W (the Wronskian) as:

$$W(t) = \det C = x_{11}(t)\ x_{22}(t) - x_{12}(t)\ x_{21}(t) \tag{34}$$

Taking the time derivative of W and using eq.(30) gives that $\dfrac{dW}{dt} = 0$, which implies that $W(t) =$ constant $= W(0) = 1$. Thus eq.(33) can be written:

$$\lambda^2 - (\mathrm{tr}C)\lambda + 1 = 0 \tag{35}$$

which has the solution:

$$\lambda = \frac{\mathrm{tr}C \pm \sqrt{\mathrm{tr}C^2 - 4}}{2} \tag{36}$$

Floquet theory showed that instability results if either eigenvalue has modulus larger than unity.

Thus if $|\mathrm{tr}C| > 2$, then (36) gives real roots. But the product of the roots is unity, so if one root has modulus less than unity, the other has modulus greater than unity, with the result that this case is UNSTABLE and corresponds to exponential growth in time.

On the other hand, if $|\mathrm{tr}C| < 2$, then (36) gives a pair of complex conjugate roots. But since their product must be unity, they must both lie on the unit circle, with the result that this case is STABLE. Note that the stability here is neutral stability not asymptotic stability, since Hill's eq.(29) has no damping. This case corresponds to quasiperiodic behavior in time.

Thus the transition from stable to unstable corresponds to those parameter values which give $|\mathrm{tr}C| = 2$. From (36), if $\mathrm{tr}C = 2$ then $\lambda = 1, 1$, and from eq.(28) this corresponds to a periodic solution with period T. On the other hand, if $\mathrm{tr}C = -2$ then $\lambda = -1, -1$, and from eq.(28) this corresponds to a periodic solution with period $2T$. This gives the important result that *on the transition curves in parameter space between stable and unstable, there exist periodic motions of period T or $2T$.*

The theory presented in this section can be used as a practical numerical procedure for determining stability of a Hill's equation. Begin by numerically integrating the o.d.e. for the two initial conditions (31). Carry each numerical integration out to time $t = T$ and so obtain $\mathrm{tr}C = x_{11}(T) + x_{22}(T)$. Then $|\mathrm{tr}C| > 2$ is unstable, while $|\mathrm{tr}C| < 2$ is stable. Note that this approach allows you to draw conclusions about large time behavior after numerically integrating for only one forcing period. Without Floquet theory you would have to numerically integrate out to large time in order to determine if a solution was growing unbounded, especially for systems which are close to a transition curve, in which case the asymptotic growth is very slow.

The reader is referred to "Nonlinear Vibrations in Mechanical and Electrical Systems" by J.Stoker, Wiley, 1950, Chapter 6, for a brief treatment of Floquet theory and Hill's equation. See "Hill's Equation" by W.Magnus and S.Winkler, Dover, 1979 for a complete treatment.

Harmonic Balance

In this section we apply Floquet theory to Mathieu's equation (1). Since the period of the forcing function in (1) is $T = 2\pi$, we may apply the result obtained in the previous section to conclude that on the transition curves in the δ-ϵ parameter plane there exist solutions of period 2π or 4π. This motivates us to look for such a solution in the form of a Fourier series:

$$x(t) = \sum_{n=0}^{\infty} a_n \cos \frac{nt}{2} + b_n \sin \frac{nt}{2} \tag{37}$$

This series represents a general periodic function with period 4π, and includes functions with period 2π as a special case (when a_{odd} and b_{odd} are zero). Substituting (37) into Mathieu's equation (1), simplifying the trig and collecting terms (a procedure called *harmonic balance)* gives four sets of algebraic equations on the coefficients a_n and b_n. Each set deals exclusively with a_{even}, b_{even}, a_{odd} and b_{odd}, respectively. Each set is homogeneous and of infinite order, so for a nontrivial solution the determinants must vanish. This gives four infinite determinants (called Hill's determinants):

$$a_{even}: \qquad \begin{vmatrix} \delta & \epsilon/2 & 0 & 0 & \\ \epsilon & \delta-1 & \epsilon/2 & 0 & \cdots \\ 0 & \epsilon/2 & \delta-4 & \epsilon/2 & \\ & & \cdots & & \end{vmatrix} = 0 \tag{38}$$

$$b_{even}: \qquad \begin{vmatrix} \delta-1 & \epsilon/2 & 0 & 0 & \\ \epsilon/2 & \delta-4 & \epsilon/2 & 0 & \cdots \\ 0 & \epsilon/2 & \delta-9 & \epsilon/2 & \\ & & \cdots & & \end{vmatrix} = 0 \tag{39}$$

$$a_{odd}: \qquad \begin{vmatrix} \delta-1/4+\epsilon/2 & \epsilon/2 & 0 & 0 & \\ \epsilon/2 & \delta-9/4 & \epsilon/2 & 0 & \cdots \\ 0 & \epsilon/2 & \delta-25/4 & \epsilon/2 & \\ & & \cdots & & \end{vmatrix} = 0 \tag{40}$$

$$b_{odd}: \qquad \begin{vmatrix} \delta-1/4-\epsilon/2 & \epsilon/2 & 0 & 0 & \\ \epsilon/2 & \delta-9/4 & \epsilon/2 & 0 & \cdots \\ 0 & \epsilon/2 & \delta-25/4 & \epsilon/2 & \\ & & \cdots & & \end{vmatrix} = 0 \tag{41}$$

In all four determinants the typical row is of the form:

$$\cdots \qquad 0 \qquad \epsilon/2 \qquad \delta - n^2/4 \qquad \epsilon/2 \qquad 0 \qquad \cdots$$

(except for the first one or two rows).

Each of these four determinants represents a functional relationship between δ and ϵ, which plots as a set of transition curves in the δ-ϵ plane. By setting $\epsilon = 0$ in these determinants it is easy to see where the associated curves intersect the δ axis. The transition curves obtained from the a_{even} and b_{even} determinants intersect the δ axis at $\delta = n^2$, $n = 0, 1, 2, \cdots$, while those obtained from the a_{odd} and b_{odd} determinants intersect the δ axis at $\delta = \dfrac{(2n+1)^2}{4}$, $n = 0, 1, 2, \cdots$. For $\epsilon > 0$, each of these points on the δ axis gives rise to two transition curves, one coming from the associated a determinant, and the other from the b determinant. Thus there is a tongue of instability emanating from each of the following points on the δ axis:

$$\delta = \frac{n^2}{4}, \qquad n = 0, 1, 2, 3, \cdots \tag{42}$$

The $n = 0$ case is an exception as only one transition curve emanates from it, as a comparison of eq.(38) with eq.(39) will show.

Note that the transition curves (19) found earlier in this Chapter by using the two variable expansion method correspond to $n = 1$ in eq.(42). Why did the perturbation method miss the other tongues of instability? It was because we truncated the perturbation method, neglecting terms of $O(\epsilon^2)$. The other tongues of instability turn out to emerge at higher order truncations in the various perturbation methods (two variable expansion, averaging, Lie transforms, normal forms, even regular perturbations). In all cases these methods deliver an expression for a particular transition curve in the form of a power series expansion:

$$\delta = \frac{n^2}{4} + \delta_1\epsilon + \delta_2\epsilon^2 + \cdots \tag{43}$$

As an alternative method of obtaining such an expansion, we can simply substitute (43) into any of the determinants (38)-(41) and collect terms, in order to obtain values for the coefficients δ_i. As an example, let us substitute (43) for $n = 1$ into the a_{odd} determinant (40). Expanding a 3×3 truncation of (40), we get (using computer algebra):

$$-\frac{\epsilon^3}{8} - \frac{\delta\,\epsilon^2}{2} + \frac{13\,\epsilon^2}{8} + \frac{\delta^2\,\epsilon}{2} - \frac{17\,\delta\,\epsilon}{4} + \frac{225\,\epsilon}{32} + \delta^3 - \frac{35\,\delta^2}{4} + \frac{259\,\delta}{16} - \frac{225}{64} \tag{44}$$

Substituting (43) with $n = 1$ into (44) and collecting terms gives:

$$(12\,\delta_1 + 6)\,\epsilon + \frac{\left(24\,\delta_2 - 16\,{\delta_1}^2 - 8\,\delta_1 + 3\right)\,\epsilon^2}{2} + \cdots \tag{45}$$

Requiring the coefficients of ϵ and ϵ^2 in (45) to vanish gives:

$$\delta_1 = -\frac{1}{2}, \qquad \delta_2 = -\frac{1}{8} \tag{46}$$

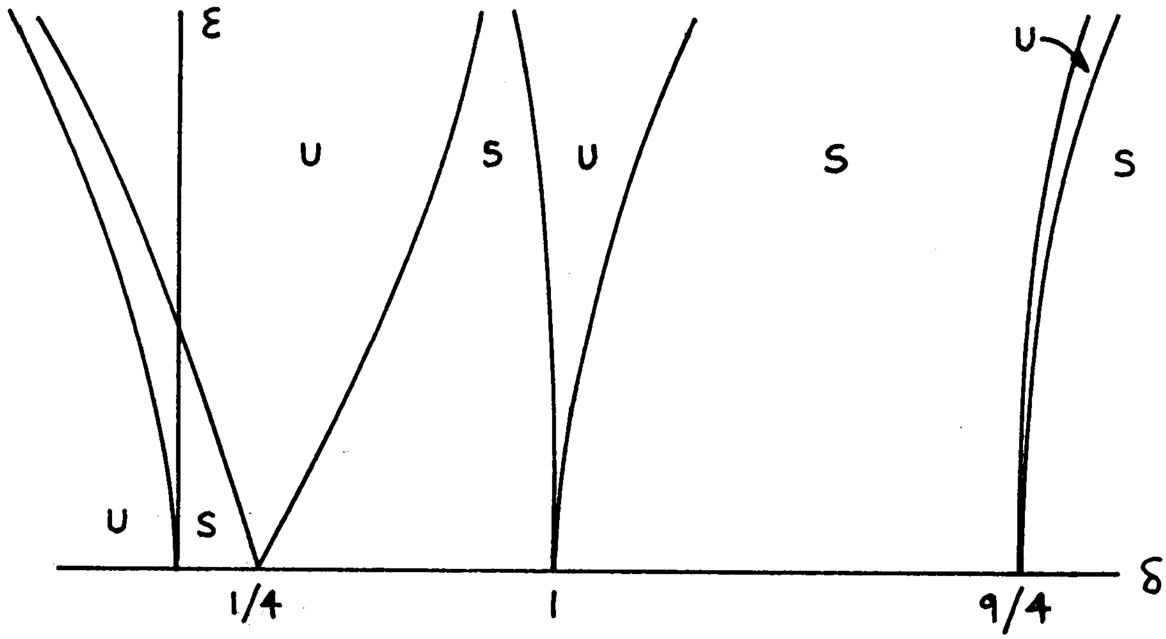
This process can be continued to any order of truncation. Here are the expansions of the first few transition curves:

$$\delta = -\frac{\epsilon^2}{2} + \frac{7\,\epsilon^4}{32} - \frac{29\,\epsilon^6}{144} + \frac{68687\,\epsilon^8}{294912} - \frac{123707\,\epsilon^{10}}{409600} + \frac{8022167579\,\epsilon^{12}}{19110297600} + \cdots \tag{47}$$

$$\begin{aligned} \delta \;=\; & \frac{1}{4} - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \frac{\epsilon^3}{32} - \frac{\epsilon^4}{384} - \frac{11\,\epsilon^5}{4608} + \frac{49\,\epsilon^6}{36864} - \frac{55\,\epsilon^7}{294912} - \frac{83\,\epsilon^8}{552960} \\ & + \frac{12121\,\epsilon^9}{117964800} - \frac{114299\,\epsilon^{10}}{6370099200} - \frac{192151\,\epsilon^{11}}{15288238080} + \frac{83513957\,\epsilon^{12}}{8561413324800} + \cdots \end{aligned} \tag{48}$$

$$\begin{aligned} \delta \;=\; & \frac{1}{4} + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} - \frac{\epsilon^3}{32} - \frac{\epsilon^4}{384} + \frac{11\,\epsilon^5}{4608} + \frac{49\,\epsilon^6}{36864} + \frac{55\,\epsilon^7}{294912} - \frac{83\,\epsilon^8}{552960} \\ & - \frac{12121\,\epsilon^9}{117964800} - \frac{114299\,\epsilon^{10}}{6370099200} + \frac{192151\,\epsilon^{11}}{15288238080} + \frac{83513957\,\epsilon^{12}}{8561413324800} + \cdots \end{aligned} \tag{49}$$

$$\begin{aligned} \delta \;=\; & 1 - \frac{\epsilon^2}{12} + \frac{5\,\epsilon^4}{3456} - \frac{289\,\epsilon^6}{4976640} + \frac{21391\,\epsilon^8}{7166361600} \\ & - \frac{2499767\,\epsilon^{10}}{14447384985600} + \frac{1046070973\,\epsilon^{12}}{97086427103232000} + \cdots \end{aligned} \tag{50}$$



Transition curves in Mathieu's equation. S=stable, U=unstable.

$$\delta = 1 + \frac{5\,\epsilon^2}{12} - \frac{763\,\epsilon^4}{3456} + \frac{1002401\,\epsilon^6}{4976640} - \frac{1669068401\,\epsilon^8}{7166361600}$$
$$+\frac{4363384401463\,\epsilon^{10}}{14447384985600} - \frac{40755179450909507\,\epsilon^{12}}{97086427103232000} + \cdots \tag{51}$$

Effect of Damping

In this section we investigate the effect that damping has on the transition curves of Mathieu's equation by applying the two variable expansion method to the following equation, known as the damped Mathieu equation:

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + (\delta + \epsilon\cos t)\; x = 0 \tag{52}$$

In order to facilitate the perturbation method, we scale the damping coefficient c to be $O(\epsilon)$:

$$c = \epsilon\mu \tag{53}$$

We can use the same setup that we did earlier in this Chapter, whereupon eq.(6) becomes:

$$\frac{\partial^2 x}{\partial\xi^2} + 2\epsilon\frac{\partial^2 x}{\partial\xi\partial\eta} + \epsilon^2\frac{\partial^2 x}{\partial\eta^2} + \epsilon\mu\left(\frac{\partial x}{\partial\xi} + \epsilon\frac{\partial x}{\partial\eta}\right) + (\delta + \epsilon\cos\xi)\; x = 0 \tag{54}$$

Now we expand x as in eq.(7) and δ as in eq.(16), and we find that eq.(17) gets an additional term:

$$\frac{\partial^2 x_1}{\partial\xi^2} + \frac{1}{4}\; x_1 = -2\frac{\partial^2 x_0}{\partial\xi\partial\eta} - x_0\cos\xi - \delta_1 x_0 - \mu\frac{\partial x_0}{\partial\xi} \tag{55}$$

which results in two additional terms appearing in the slow flow eqs.(18):

$$\frac{dA}{d\eta} = -\frac{\mu}{2}A + \left(\delta_1 - \frac{1}{2}\right)B, \qquad \frac{dB}{d\eta} = -\left(\delta_1 + \frac{1}{2}\right)A - \frac{\mu}{2}B \tag{56}$$

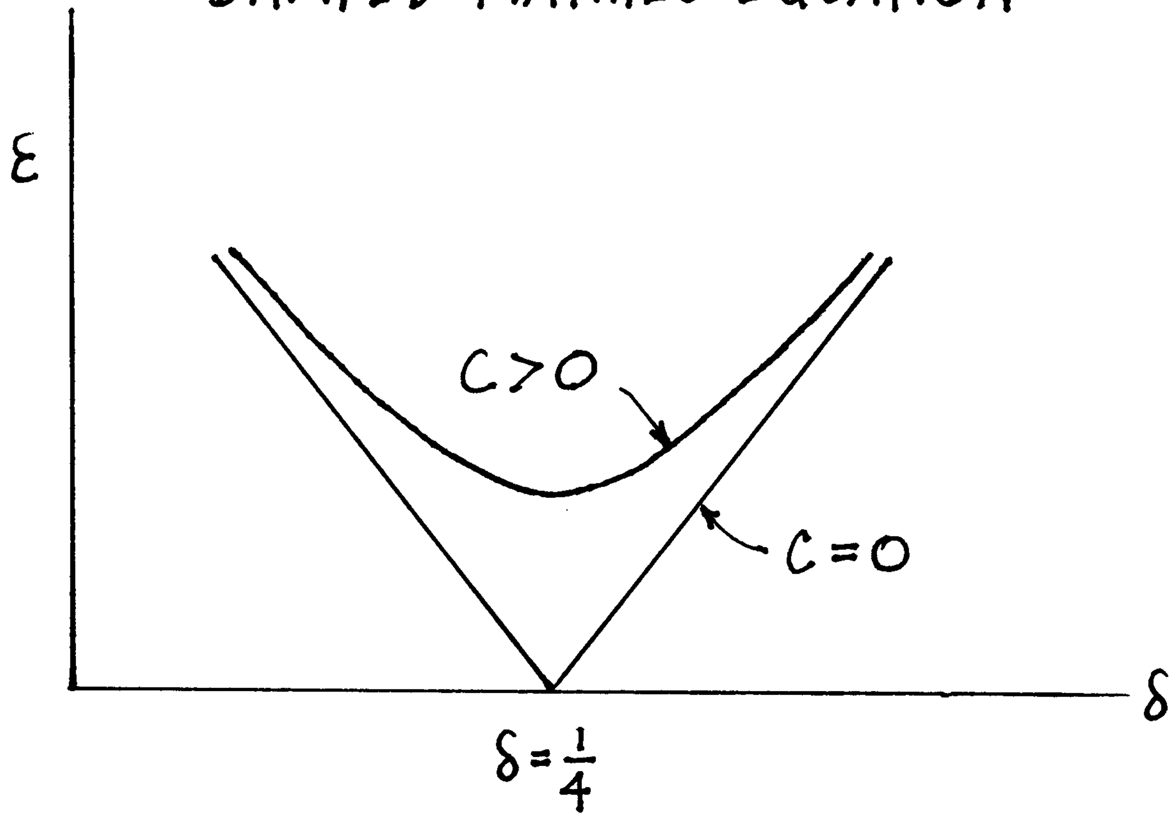
Eqs.(56) are a linear constant coefficient system which may be solved by assuming a solution in the form $A(\eta) = A_0\exp(\lambda\eta)$, $B(\eta) = B_0\exp(\lambda\eta)$. For nontrivial constants A_0 and B_0, the following determinant must vanish:

$$\begin{vmatrix} -\frac{\mu}{2} - \lambda & -\frac{1}{2} + \delta_1 \\ -\frac{1}{2} - \delta_1 & -\frac{\mu}{2} - \lambda \end{vmatrix} = 0 \qquad \Rightarrow \qquad \lambda = -\frac{\mu}{2} \pm \sqrt{-\delta_1^2 + \frac{1}{4}} \tag{57}$$

For the transition between stable and unstable, we set $\lambda = 0$, giving the following value for δ_1:

$$\delta_1 = \pm\frac{\sqrt{1-\mu^2}}{2} \tag{58}$$

DAMPED MATHIEU EQUATION



This gives the following expressions for the $n = 1$ transition curves:

$$\delta = \frac{1}{4} \pm \epsilon \frac{\sqrt{1-\mu^2}}{2} + O(\epsilon^2) = \frac{1}{4} \pm \frac{\sqrt{\epsilon^2 - c^2}}{2} + O(\epsilon^2) \tag{59}$$

Eq.(59) predicts that for a given value of c there is a minimum value of ϵ which is required for instability to occur. The $n = 1$ tongue, which for $c = 0$ emanates from the δ axis, becomes detached from the δ axis for $c > 0$. This prediction is verified by numerically integrating eq.(52) for fixed c, while δ and ϵ are permitted to vary.

Effect of Nonlinearity

In the previous sections of this Chapter we have seen how unbounded solutions to Mathieu's equation (1) can result from resonances between the forcing frequency and the oscillator's unforced natural frequency. However, real physical systems do not exhibit unbounded behavior. The difference lies in the fact that the Mathieu equation is linear. The effects of nonlinearity can be explained as follows: as the resonance causes the amplitude of the motion to increase, the relation between period and amplitude (which is a characteristic effect of nonlinearity) causes the resonance to detune, decreasing its tendency to produce large motions.

A more realistic model can be obtained by including nonlinear terms in the Mathieu equation. For example, in the case of the vertically driven pendulum, eq.(2), if we expand $\sin x$ in a Taylor series, we get:

$$\frac{d^2x}{dt^2} + \left(\frac{g}{L} - \frac{A\omega^2}{L}\cos\omega t\right)\left(x - \frac{x^3}{6} + \cdots\right) = 0 \tag{60}$$

Now if we rescale time by $\tau = \omega t$ and set $\delta = \dfrac{g}{\omega^2 L}$ and $\epsilon = \dfrac{A}{L}$, we get:

$$\frac{d^2x}{d\tau^2} + (\delta - \epsilon\cos\tau)\left(x - \frac{x^3}{6} + \cdots\right) = 0 \tag{61}$$

Next, if we scale x by $x = \sqrt{\epsilon}\, y$ and neglect terms of $O(\epsilon^2)$, we get:

$$\frac{d^2y}{d\tau^2} + (\delta - \epsilon\cos\tau)\, y - \epsilon\frac{\delta}{6}y^3 + O(\epsilon^2) = 0 \tag{62}$$

Motivated by this example, in this section we study the following nonlinear Mathieu equation:

$$\frac{d^2x}{dt^2} + (\delta + \epsilon\cos t)\, x + \epsilon\alpha x^3 = 0 \tag{63}$$

We once again use the two variable expansion method to treat this equation. Using the same setup that we did earlier in this Chapter, eq.(6) becomes:

$$\frac{\partial^2 x}{\partial\xi^2} + 2\epsilon\frac{\partial^2 x}{\partial\xi\partial\eta} + \epsilon^2\frac{\partial^2 x}{\partial\eta^2} + (\delta + \epsilon\cos\xi)\; x + \epsilon\alpha x^3 = 0 \tag{64}$$

We expand x as in eq.(7) and δ as in eq.(16), and we find that eq.(17) gets an additional term:

$$\frac{\partial^2 x_1}{\partial \xi^2} + \frac{1}{4}\, x_1 = -2\frac{\partial^2 x_0}{\partial \xi \partial \eta} - x_0 \cos\xi - \delta_1 x_0 - \alpha x_0^3 \tag{65}$$

where x_0 is of the form:

$$x_0(\xi,\eta) = A(\eta)\cos\frac{\xi}{2} + B(\eta)\sin\frac{\xi}{2} \tag{66}$$

Removal of resonant terms in (65) results in the appearance of some additional cubic terms in the slow flow eqs.(18):

$$\frac{dA}{d\eta} = \left(\delta_1 - \frac{1}{2}\right) B + \frac{3\alpha}{4} B(A^2+B^2), \qquad \frac{dB}{d\eta} = -\left(\delta_1 + \frac{1}{2}\right) A - \frac{3\alpha}{4} A(A^2+B^2) \tag{67}$$

In order to more easily work with the slow flow (67), we transform to polar coordinates in the A-B phase plane:

$$A = R\ \cos\theta, \qquad B = R\ \sin\theta \tag{68}$$

Note that eqs.(68) and (66) give the following alternate expression for x_0:

$$x_0(\xi,\eta) = R(\eta)\cos\left(\frac{\xi}{2} - \theta(\eta)\right) \tag{69}$$

Substitution of (68) into the slow flow (67) gives:

$$\frac{dR}{d\eta} = -\frac{R}{2}\ \sin 2\theta, \qquad \frac{d\theta}{d\eta} = -\delta_1 - \frac{\cos 2\theta}{2} - \frac{3\alpha}{4} R^2 \tag{70}$$

We seek equilibria of the slow flow (70). From (69), a solution in which R and θ are constant in slow time η represents a periodic motion of the nonlinear Mathieu equation (63) which has one-half the frequency of the forcing function, that is, such a motion is a 2:1 subharmonic. Such slow flow equilibria satisfy the equations:

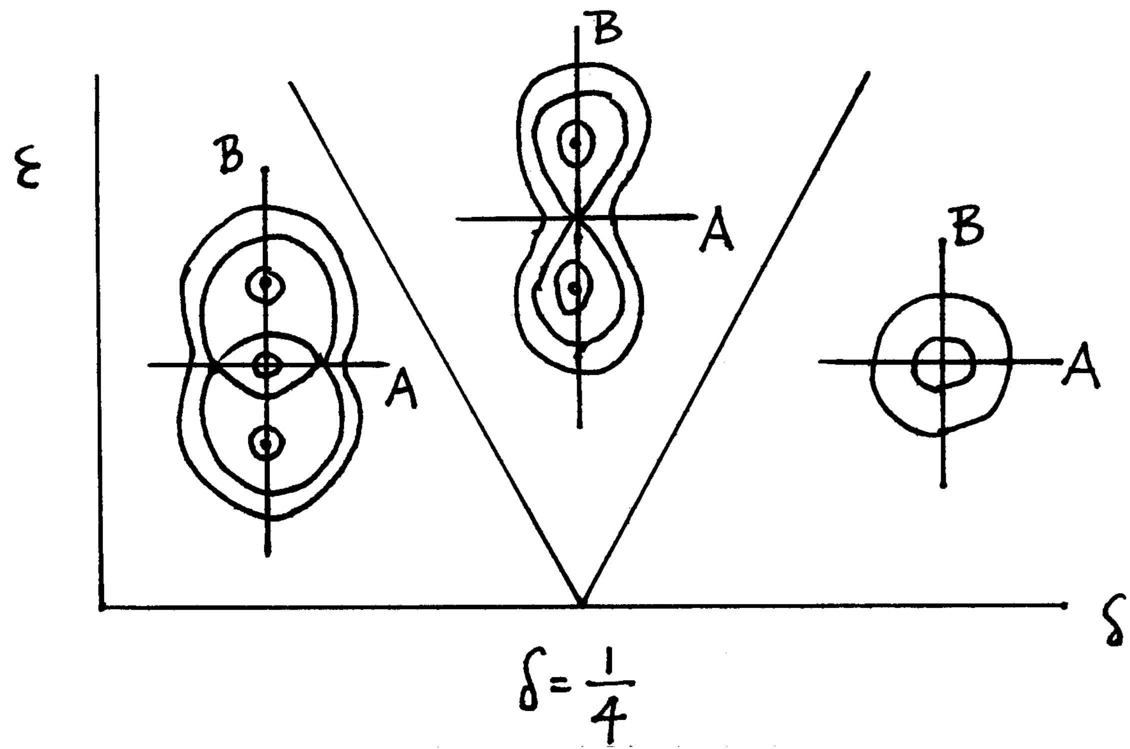
$$-\frac{R}{2}\ \sin 2\theta = 0, \qquad -\delta_1 - \frac{\cos 2\theta}{2} - \frac{3\alpha}{4} R^2 = 0 \tag{71}$$

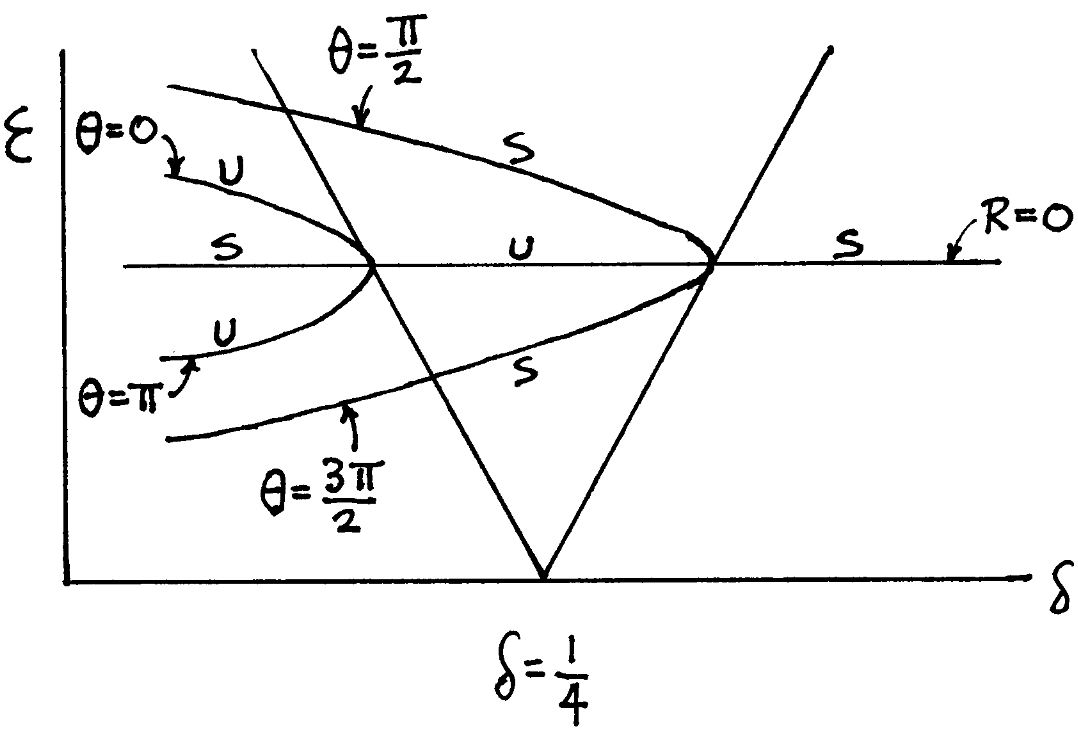
Ignoring the trivial solution $R = 0$, the first eq. of (71) requires $\sin 2\theta = 0$ or $\theta = 0, \frac{\pi}{2}, \pi$ or $\frac{3\pi}{2}$. Solving the second eq. of (71) for R^2, we get:

$$R^2 = -\frac{4}{3\alpha}\left(\frac{\cos 2\theta}{2} + \delta_1\right) \tag{72}$$

For a nontrivial real solution, $R^2 > 0$. Let us assume that the nonlinearity parameter $\alpha > 0$. Then in the case of $\theta = 0$ or π, $\cos 2\theta = 1$ and nontrivial equilibria exist only for $\delta_1 < -\frac{1}{2}$. On the other hand, for $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, $\cos 2\theta = -1$ and nontrivial equilibria require $\delta_1 < \frac{1}{2}$.

Nonlinear Mathieu Equation ($\alpha > 0$)





Since $\delta_1 = \pm\dfrac{1}{2}$ corresponds to transition curves for the stability of the trivial solution, the analysis predicts that bifurcations occur as we cross the transition curves in the δ-ϵ plane. That is, imagine quasistatically decreasing the parameter δ while ϵ is kept fixed, and moving through the $n = 1$ tongue emanating from the point $\delta = \dfrac{1}{4}$ on the δ axis. As δ decreases across the right transition curve, the trivial solution $x = 0$ becomes unstable and simultaneously a stable 2:1 subharmonic motion is born. This motion grows in amplitude as δ continues to decrease. When the left transition curve is crossed, the trivial solution becomes stable again, and an unstable 2:1 subharmonic is born. This scenario can be pictured as involving two pitchfork bifurcations.

If the nonlinearity parameter $\alpha < 0$, a similar sequence of bifurcations occurs, except in this case the subharmonic motions are born as δ *increases* quasistatically through the $n = 1$ tongue.

Problems

Problem 6.1

Alternatives to Floquet theory. As we saw in this Chapter, Floquet theory offers an approach to determining the stablity (that is the boundedness of all solutions) of the n-dimensional linear system with periodic coefficients:

$$\frac{dx}{dt} = A(t)\, x, \qquad A(t+T) = A(t) \tag{73}$$

where x is an n-vector and $A(t)$ is an $n \times n$ matrix.

This problem involves three alternative approaches. For each one, decide whether or not it is valid. If you think a method is valid, offer a line of reasoning showing why it works. If you think it is wrong, explain why it doesn't work or find a counterexample.

1. Set $x = Ty$ where y is an n-vector and T is an $n \times n$ matrix. Then $\frac{dy}{dt} = T^{-1}ATy$. Choose T such that $T^{-1}AT = D$ is diagonal (or more generally in Jordan canonical form). Then study the uncoupled system $\frac{dy}{dt} = Dy$.

2. Consider $\frac{dx}{dt} = A(t^*)\, x$ for t^* a fixed value of t. Examine the eigenvalues of $A(t^*)$. If the real parts of these eigenvalues remain negative for all positive t^*, then the solutions are asymptotically stable.

3. Replace the given equations by the averaged equations, $\frac{dx}{dt} = B\, x$, where $B = \frac{1}{T}\int_0^T A(t)dt$. Note that B is a constant coefficient matrix. Use the usual stability criteria on $\frac{dx}{dt} = B\, x$.

Problem 6.2
Nonlinear parametric resonance. This problem concerns the following differential equation:

$$\frac{d^2x}{dt^2} + \left(\frac{1}{4} + \epsilon k_1\right) x + \epsilon x^3 \cos t = 0, \qquad \epsilon << 1 \tag{74}$$

a) Use the two variable expansion method to derive a slow flow, neglecting terms of $O(\epsilon^2)$.

b) Analyze the slow flow. In particular, determine all slow flow equilibria and their stability. Make a sketch of the slow flow phase portrait for $k_1 = 0$ and for $k_1 = 0.1$.

Problem 6.3
The particle in the plane. Earlier in this Chapter we showed that the stability of the x-mode of the particle in the plane is governed by eq.(4) which may be written in the form:

$$\frac{d^2v}{dt^2} + \left(\frac{\delta - \epsilon \cos^2 t}{1 - \epsilon \cos^2 t}\right) v = 0 \tag{75}$$

where $\delta = 1 - L$ and $\epsilon = A^2$. Using the method of harmonic balance, obtain an approximate expression for the transition curve in the δ-ϵ plane which passes through the origin ($\delta = 0,\ \epsilon = 0$). Neglect terms of $O(\epsilon^4)$.

Problem 6.4
Damped Mathieu equation and Floquet theory. This question concerns eq.(52) for δ=1/4, exact 2:1 resonance (no detuning):

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + (\frac{1}{4} + \epsilon \cos t)\ x = 0 \tag{76}$$

a. Find an approximate expression for the transition curve separating stable regions from unstable regions in the c-ϵ parameter plane, valid for small ϵ.
b. Compare your answer with results obtained by numerically integrating eq.(76) in conjunction with Floquet theory.
Hint: For a given pair of parameters (c, ϵ), numerically integrate (76) twice, respectively for initial conditions x=1, dx/dt=0 and x=0, $dx/dt = 1$. Evaluate the two resulting solution vectors at time $t = 2\pi$, and use them as the columns in the fundamental solution matrix $X(T)$ referred to in eq.(22). Compute the eigenvalues λ_1, λ_2 of this matrix. As discussed in the text, stability requires that both eigenvalues satisfy $|\lambda_i| < 1$.