

Bifurcation of nonlinear normal modes in a class of two degree of freedom systems

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Summary. This work concerns the nonlinear vibrations of a class of two-degree-of-freedom autonomous conservative systems consisting of two coupled nonlinear oscillators with cubic coupling forces. For such a system, nonlinear normal modes (NNM's) have long been studied as "vibrations in unison", i.e. periodic motions in which x and y simultaneously achieve zero velocity. We investigate the stability and bifurcation of NNM's by using a perturbation method together with the computer algebra system MACSYMA.

The results of our study include the characterization of those systems which are maximally degenerate in the sense of changing the number of elementary periodic motions which they exhibit in response to a small change in parameters. By looking in the neighborhood of such a degenerate system, we obtain universal unfoldings. In particular we show how NNM's, which project onto the configuration plane as (curved) line segments, are related to a family of periodic orbits which look like ellipses on the configuration plane. Our analytic results are shown to compare favorably with the results of numerical integration.

1 Introduction

This work concerns a class of periodic motions (PM's) called nonlinear normal modes (NNM's). NNM's may be thought of as nonlinear continuations of the linear normal modes (LNM's) associated with eigenvalue problems in a linearized system. Interest in LNM's may be said to stem both from their role as basis vectors for the general solution of a linear system, as well as from their role as sources of large deflection of forced linear structures due to the phenomenon of resonance. Although the principle of superposition cannot be applied to nonlinear systems, early interest in NNM's focused on nonlinear resonance. NNM's were studied by Kauderer [5], Rosenberg [16] and others [4, 9, 10, 12, 13] since the 1960's. More recently, Vakakis [19] examined NNM's from the point of view of modern dynamics. It was shown [16] that NNM's can be either "similar" or "nonsimilar". Similar modes correspond to straight lines in the configuration space of the system (and in this sense they resemble LNM's), whereas nonsimilar NNM's correspond to curved lines. Methods for detecting NNM's in discrete nonlinear systems already exist in the literature [8, 14, 16]. Recently invariant manifold concepts have been utilized for this purpose [17].

In previous works [7, 9] it was shown that nonlinear forced resonance occurs in the neighborhood of NNM's. Moreover, it was shown that the bifurcations of NNM's affect the topological structure of the resonance curves in a class of systems [19], and their global dynamics [20]. The study of NNM's and their bifurcations is thus of practical importance for an understanding of the dynamics of nonlinear oscillators.

Until now, only bifurcations of NNM's to other NNM's were studied in the literature. The present work views NNM's as they are related to more general PM's. According to "Poincare's conjecture" ([18], p. 381) phase space in a compact Hamiltonian system is dense with PM's. In some nonlinear systems [2] PM's have been categorized by their topology into families which are connected by bifurcations due to changes in parameters. These branches of PM's are similar to branches of equilibria encountered in nonlinear buckling problems [1]. Our goal is to investigate how NNM's bifurcate, i.e. how they change their number and form in response to a change in parameters. We conduct our study in a general class of two degree of freedom systems involving six independent parameters. We use the two variable expansion perturbation method and we use computer algebra (MACSYMA) to handle the large quantities of algebra involved in the computations. Since the perturbation method is valid in the small amplitude limit, our results are necessarily approximate. To within this approximation, we obtain a closed form solution in the form of a first integral for the slow flow. Using this result, we determine the most degenerate singularity involving a confluence of PM's, and we obtain a universal unfolding of this singularity, thereby revealing how the nearby families of PM's are related.

2 Perturbation method

We consider the general class of two degree of freedom conservative autonomous systems with kinetic energy T :

$$T = \frac{1}{2} \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right] \quad (1)$$

and potential energy V :

$$V = V_1 + \varepsilon V_2 \quad (2)$$

where

$$V_1 = \frac{1}{2} (x^2 + \omega^2 y^2), \quad (3)$$

$$V_2 = a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4 \quad (4)$$

and where

$$\omega^2 = 1 + \varepsilon\Delta. \quad (5)$$

This system evolves according to the following two differential equations:

$$\frac{d^2x}{dt^2} + x = -\varepsilon \frac{\partial V_2}{\partial x}, \quad (6)$$

$$\frac{d^2y}{dt^2} + (1 + \varepsilon\Delta) y = -\varepsilon \frac{\partial V_2}{\partial y}, \quad (7)$$

and is characterized by the five nonlinear restoring force coefficients a_{ij} and a detuning coefficient Δ . The parameter ε is included in order to provide a scaling for the perturbation method, and so we assume $\varepsilon \ll 1$ and we neglect terms of order ε^2 throughout. Note that when $\varepsilon = 0$ the system is in 1:1 resonance so that we are perturbing off of a system in which the linear normal modes are degenerate.

In order to obtain an approximate solution for the system (1)–(7), we use the two-variable expansion method [6, 11, 15]. We replace t as independent variable by the two variables

$$\xi = t \quad \text{and} \quad \eta = \varepsilon t \quad (8)$$

in which case Eq. (6) becomes

$$\frac{\partial^2 x}{\partial \xi^2} + 2\varepsilon \frac{\partial^2 x}{\partial \xi \partial \eta} + \varepsilon^2 \frac{\partial^2 x}{\partial \eta^2} + x = -\varepsilon \frac{\partial V_2}{\partial x} \quad (9)$$

and where Eq. (7) is similarly transformed. Next we expand x and y in power series in ε ,

$$x(\xi, \eta, \varepsilon) = x_0(\xi, \eta) + \varepsilon x_1(\xi, \eta) + \dots \quad (10)$$

and a similar expression for y . Substitution into the d.e.'s on x and y (e.g. Eq. (9)) gives, after collecting terms and equating the coefficients of like powers of ε to zero:

$$\frac{\partial^2 x_0}{\partial \xi^2} + x_0 = 0, \quad \frac{\partial^2 y_0}{\partial \xi^2} + y_0 = 0, \quad (11)$$

$$\frac{\partial^2 x_1}{\partial \xi^2} + x_1 = -2 \frac{\partial^2 x_0}{\partial \xi \partial \eta} - \frac{\partial V_2}{\partial x} \Big|_{x=x_0, y=y_0} \quad (12)$$

and a similar eq. on y_1 . We take the solution of Eqs. (11) in the form:

$$x_0 = A(\eta) \cos \xi + B(\eta) \sin \xi, \quad (13)$$

$$y_0 = C(\eta) \cos \xi + D(\eta) \sin \xi \quad (14)$$

where $A(\eta)$, $B(\eta)$, $C(\eta)$, $D(\eta)$ are arbitrary functions of η . We substitute Eqs. (13), (14) into the d.e.'s on x_1 and y_1 (e.g. Eq. (12)), trigonometrically simplify and collect terms, and equate to zero the coefficients of the resonant terms $\cos \xi$ and $\sin \xi$, giving:

$$\begin{aligned} \frac{dA}{d\eta} = \frac{1}{8} [3a_{13}(C^2 + D^2) D + 3a_{31}(A^2 + 3B^2) D + 2a_{22}(C^2 + 3D^2) B \\ + (4a_{22}D + 6a_{31}B) AC + 12a_{40}(A^2 + B^2) B] \end{aligned} \quad (15)$$

and three similar eqs. on $B(\eta)$, $C(\eta)$, $D(\eta)$. Next we transform to polar coordinates, $R_1(\eta)$, $R_2(\eta)$, $\theta_1(\eta)$, $\theta_2(\eta)$

$$A = R_1 \cos \theta_1, \quad B = R_1 \sin \theta_1, \quad C = R_2 \cos \theta_2, \quad D = R_2 \sin \theta_2 \quad (16)$$

noting that the zeroth order solution, Eqs. (13), (14), can be written

$$x_0 = R_1(\eta) \cos(\xi - \theta_1(\eta)), \quad y_0 = R_2(\eta) \cos(\xi - \theta_2(\eta)). \quad (17)$$

Substituting Eqs. (16) into the d.e.'s on A, B, C, D (e.g. Eq. (15)) we obtain d.e.'s on $R_1, R_2, \theta_1, \theta_2$ which are naturally written in terms of the variable φ :

$$\varphi = \theta_2 - \theta_1, \quad (18)$$

$$\frac{dR_1}{d\eta} = \frac{1}{4} a_{22} R_1 R_2^2 \sin 2\varphi + \frac{3}{8} [a_{13} R_2^3 + a_{31} R_1^2 R_2] \sin \varphi, \quad (19)$$

$$\frac{dR_2}{d\eta} = -\frac{1}{4} a_{22} R_1^2 R_2 \sin 2\varphi - \frac{3}{8} [a_{13} R_1 R_2^2 + a_{31} R_1^3] \sin \varphi, \quad (20)$$

$$\begin{aligned} \frac{d\theta_1}{d\eta} = & -\frac{1}{4} a_{22} R_2^2 [\cos 2\varphi + 2] - \frac{3}{8} a_{13} \frac{R_2^3}{R_1} \cos \varphi \\ & - \frac{9}{8} a_{31} R_1 R_2 \cos \varphi - \frac{3}{2} a_{40} R_1^2, \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{d\theta_2}{d\eta} = & -\frac{1}{2} - \frac{1}{4} a_{22} R_1^2 [\cos 2\varphi + 2] - \frac{3}{8} a_{31} \frac{R_1^3}{R_2} \cos \varphi \\ & - \frac{9}{8} a_{13} R_1 R_2 \cos \varphi - \frac{3}{2} a_{04} R_2^2. \end{aligned} \quad (22)$$

Subtracting Eq. (21), (22) and using Eq. (18), we obtain a differential equation on $\varphi(\eta)$:

$$\begin{aligned} \frac{d\varphi}{d\eta} = & -\frac{1}{2} + \frac{3}{8} a_{13} \cos \varphi \left[\frac{R_2^3}{R_1} - 3R_1 R_2 \right] \\ & - \frac{3}{8} a_{31} \cos \varphi \left[\frac{R_1^3}{R_2} - 3R_1 R_2 \right] - \frac{3}{2} [a_{04} R_2^2 - a_{40} R_1^2] \\ & + \frac{1}{4} a_{22} [R_2^2 - R_1^2] (\cos 2\varphi + 2). \end{aligned} \quad (23)$$

Finally we transform to polars R, ψ in the $R_1 - R_2$ plane,

$$R_1 = R \cos \psi, \quad R_2 = R \sin \psi \quad (24)$$

which replaces Eqs. (19), (20), (23) by the following:

$$\frac{dR}{d\eta} = 0, \quad (25)$$

$$\frac{d\psi}{d\eta} = -\frac{1}{8} R^2 a_{22} \sin 2\varphi \sin 2\psi + \frac{3}{16} R^2 \sin \varphi [(a_{13} - a_{31}) \cos 2\psi - (a_{13} + a_{31})], \quad (26)$$

$$\begin{aligned} \frac{d\varphi}{d\eta} = & -\frac{\Delta}{2} + R^2 \left(-\frac{3}{8} a_{31} \cos \varphi [\cos^2 \psi \cot \psi - \frac{3}{2} \sin 2\psi] \right. \\ & + \frac{3}{8} a_{13} \cos \varphi \left[\sin^2 \psi \tan \psi - \frac{3}{2} \sin 2\psi \right] \\ & \left. - \frac{3}{2} [a_{04} \sin^2 \psi - a_{40} \cos^2 \psi] - \frac{1}{4} a_{22} \cos 2\psi (\cos 2\varphi + 2) \right). \end{aligned} \quad (27)$$

From Eq. (25), we see that R is a constant, a result equivalent to conservation of energy. From Eqs. (26), (27) we may obtain the single first order differential equation:

$$\frac{d\varphi}{d\psi} = \frac{f(\varphi, \psi)}{g(\varphi, \psi)} \quad (28)$$

where

$$\begin{aligned} f(\varphi, \psi) = & \left[4 \frac{\Delta}{R^2} - 2a_{22} + 12a_{04} \right] \cos \psi \sin \psi \\ & + [4a_{22} - 12(a_{40} + a_{04})] \cos^3 \psi \sin \psi \\ & + [-3a_{13} + (15a_{13} - 9a_{31}) \cos^2 \psi + 12(a_{31} - a_{13}) \cos^4 \psi] \cos \varphi \\ & + 4a_{22} \cos^2 \varphi \sin \psi \cos \psi \cos 2\psi, \end{aligned} \quad (29)$$

$$g(\varphi, \psi) = [3(a_{31} - a_{13}) \cos^2 \psi + 3a_{13} + 4a_{22} \cos \varphi \cos \psi \sin \psi] \sin \varphi \cos \psi \sin \psi. \quad (30)$$

In order to integrate Eq. (28), we write it in the form $g d\varphi - f d\psi = 0$, which turns out to be an exact differential. This leads us to the following first integral of Eqs. (28)–(30):

$$\begin{aligned} & -\frac{\Delta}{R^2} \cos 2\psi + \frac{3}{8} (a_{31} - a_{13}) \cos \varphi \sin 4\psi + \frac{3}{4} (a_{31} + a_{13}) \cos \varphi \sin 2\psi \\ & + \frac{3}{8} (a_{40} + a_{04}) \cos 4\psi + \frac{3}{2} (a_{40} - a_{04}) \cos 2\psi + \frac{1}{4} a_{22} \cos^2 \varphi \\ & - \frac{1}{8} a_{22} [2 + \cos 2\varphi] \cos 4\psi = \text{constant}. \end{aligned} \quad (31)$$

For a given set of parameters a_{ij} and Δ/R^2 , we may plot the integral curves of Eq. (31) on the $\psi - \varphi$ phase plane, for $0 < \psi < \pi/2$, $0 \leq \varphi \leq 2\pi$. We exclude the line $\psi = 0$ since on this line R_2 vanishes and θ_2 is not defined. Similarly we exclude the line $\psi = \pi/2$ since on this line R_1 vanishes and θ_1 is not defined. In fact, the motion $R_2 \equiv 0$ corresponds to the “ x -mode”, $y \equiv 0$, which exists iff $a_{31} = 0$. Similarly, the motion $R_1 \equiv 0$ corresponds to the “ y -mode”, $x \equiv 0$, and it

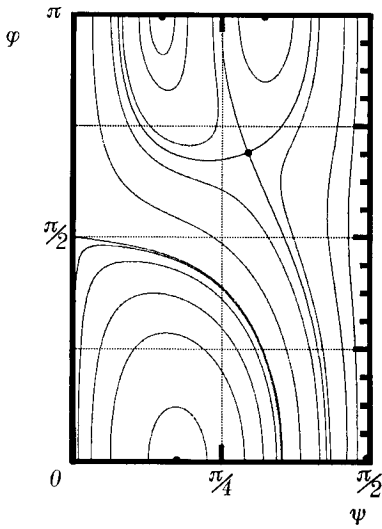


Fig. 1. Approximate first integral, Eq. (31), for parameters (32)

exists iff $a_{13} = 0$. Thus these special PM's (which are "similar" NNM's) may be investigated directly from Eqs. (6), (7) and will not be considered in our perturbation result Eq. (31).

Note that the first integral (31) is invariant under the transformation $\varphi \rightarrow 2\pi - \varphi$, and thus the phase portrait in the region $\pi \leq \varphi \leq 2\pi$ is, when reflected about the line $\varphi = \pi$, the mirror image of that in $0 \leq \varphi \leq \pi$. Therefore we may display our results in the region $0 < \psi < \pi/2$, $0 \leq \varphi \leq \pi$. See Fig. 1, where we display Eq. (31) for the parameter values:

$$a_{13} = 0, \quad a_{22} = 1, \quad a_{31} = 1, \quad a_{04} = 0, \quad a_{40} = \frac{1}{4}, \quad \Delta/R^2 = 0. \tag{32}$$

In the next section we will discuss the significance of some of the details in Fig. 1.

3 Periodic motions

Examination of Fig. 1 shows the presence of singular points. Such points represent solutions at which φ and ψ are fixed in time, which, from (24), (21), (22) and (17) mean that the amplitudes R_1 and R_2 are fixed, the phases θ_1 and θ_2 are linear in slow time η , and the phase difference is fixed between the x and y motions. Thus to $O(\varepsilon)$ such motions represent PM's of the system (1)–(7). In the case that φ is 0 or π , the x and y motions represent vibrations-in-unison, both motions simultaneously achieving their maximum displacements and velocities. These PM's plot as a single-valued curve $y = y(x)$ in the $x - y$ configuration plane (approximately a straight line for small ε). Such motions were defined by Rosenberg [16] as NNM's.

In contrast to NNM's, a singular point in Fig. 1 which corresponds to a value of φ which is unequal to 0 or π represents a PM which is not a NNM. To investigate the form of such PM's, we rewrite Eq. (17.2) using Eq. (18):

$$y_0 = R_2 \cos(\xi - \theta_1 - \varphi) = R_2[\cos(\xi - \theta_1) \cos \varphi + \sin(\xi - \theta_1) \sin \varphi] \tag{33}$$

which becomes, using Eq. (17.1),

$$y_0 = R_2 \left[\frac{x_0}{R_1} \cos \varphi + \sqrt{1 - \left(\frac{x_0}{R_1}\right)^2} \sin \varphi \right]. \tag{34}$$

Eq. (34) may be simplified so that it takes the form of a conic section:

$$[x_0 \ y_0] \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = 1 \quad (35)$$

where

$$\begin{aligned} k_{11} &= (R_1 \sin \varphi)^{-2}, \\ k_{12} = k_{21} &= -(R_1 R_2 \tan \varphi \sin \varphi)^{-1}, \\ k_{22} &= (R_2 \sin \varphi)^{-2}. \end{aligned} \quad (36)$$

The trace and determinant of the matrix $[k_{ij}]$ are positive for such PM's:

$$\text{trace} = \left(\frac{1}{R_1^2} + \frac{1}{R_2^2} \right) \frac{1}{\sin^2 \varphi} > 0, \quad (37)$$

$$\text{determinant} = [R_1 R_2 \sin \varphi]^{-2} > 0. \quad (38)$$

Thus the eigenvalues of $[k_{ij}]$ are real and positive, and the orbit of such a PM is an ellipse on the $x - y$ plane. We shall denote such a PM as an elliptic orbit (EO). EO's correspond to singularities in the $\varphi - \psi$ plane for which $\varphi \neq 0, \pi$, cf. Fig. 1.

Note that although both $\psi = 0$ and $\psi = \pi/2$ are integral curves in Fig. 1, they respectively correspond to the vanishing of R_2 and R_1 , at which points θ_2 and θ_1 are respectively undefined. In order to investigate the existence of motions corresponding to the vanishing of R_2 or R_1 , we proceed directly from Eqs. (6), (7). These show that since $a_{13} = 0$, the y -mode, $x \equiv 0$ is a PM. However, since $a_{31} = 1$, the x -mode, $y \equiv 0$, is not a PM of the system. We conclude that on the lines $\psi = 0, \pi/2$ the preceding analysis gives spurious results and is inapplicable. In particular the singular point found on the line $\psi = 0$ in Fig. 1 must not be interpreted as a PM.

In addition to revealing the existence of PM's, the first integral (31) also discloses their stability. The stability of the PM in the original system (6), (7) is the same as the stability of the singular point in the slow flow (26), (27). Since the system is conservative, only centers and saddles can generically occur, and so their stability is easily discerned. E.g. in Fig. 1, the three NNM's are stable while the EO is unstable.

An approximation for the frequency p of a PM may be obtained from Eq. (17) as follows: First we find the associated values of φ and ψ corresponding to a singularity of Eqs. (26), (27). Then for a given value of R , Eq. (24) yields R_1 and R_2 . Substitution of these values into Eqs. (21), (22) gives (equal) values for $\frac{d\theta_1}{d\eta} = \frac{d\theta_2}{d\eta} = k$ (say) which may be integrated to give

$$\theta_1 = k\eta + \theta_{10} \quad \text{and} \quad \theta_2 = k\eta + \theta_{20}. \quad (39)$$

Then from Eqs. (17) we find the frequency p as:

$$p = 1 + \varepsilon k. \quad (40)$$

4 Numerical simulations

In order to verify the analytical results of the perturbation analysis, we numerically integrated the nonlinear equations of motion (6), (7), using a fourth-order Runge-Kutta algorithm. First, the system with parameters of Eq. (32) was examined, with $\varepsilon = 0.1$ and $R = 1$, cf. Fig. 1. The equations of motion were integrated with initial conditions corresponding to the analytically predicted periodic motions. Figure 2 shows the result for the EO of Fig. 1 which corresponds to $\varphi = 2.17$ and $\psi = 0.923$. The orbital instability of this motion is verified by the projection on the configuration plane of Fig. 2a, and the time response of $x(t)$ in Fig. 2b. All the analytically predicted NNM's were verified, and their orbital stability was confirmed. As an example, Fig. 3 shows the NNM at $\varphi = 0$, $\psi = 0.54$. In the configuration plane of Fig. 3a the motion is approximately a straight line passing through the origin (in accordance with theory), and the time response of $x(t)$ in Fig. 3b contains only small amplitude modulation.

As an example of a system with a stable EO, the following parameters were chosen:

$$a_{13} = 1, \quad a_{22} = 1.4, \quad a_{31} = 0.8, \quad a_{04} = 2.2, \quad a_{40} = 0.5, \quad \Delta = -0.1, \quad R = 0.2. \quad (41)$$

The perturbation result (31) is displayed in Fig. 4. The EO at $\varphi = 2.88$, $\psi = 0.799$ is examined in Fig. 5 for $\varepsilon = 0.1$, verifying its orbital stability. Note the shape of the EO in the configuration

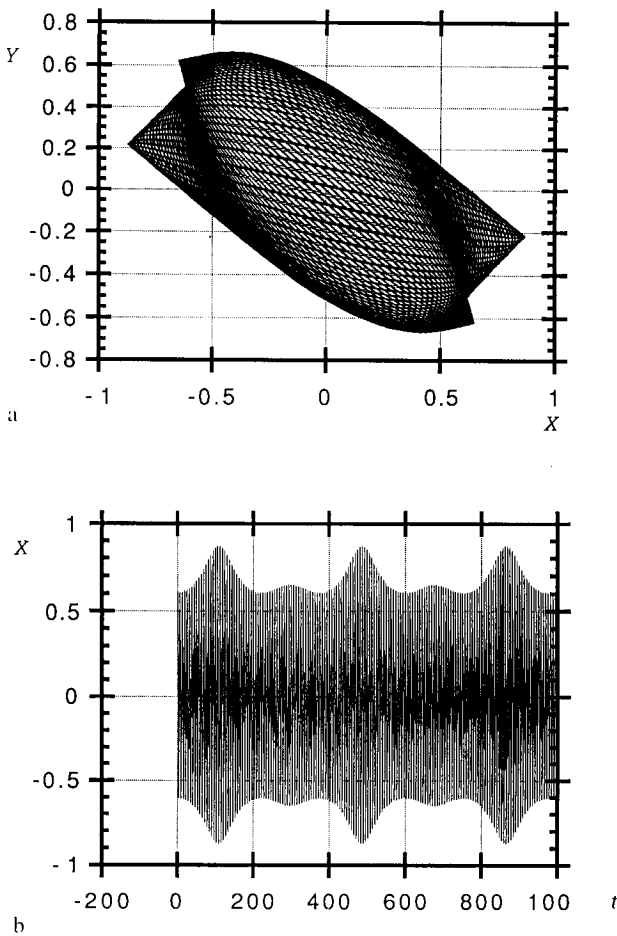


Fig. 2. Numerical integration of Eqs. (6), (7) for parameters (32) and $\varepsilon = 0.1$, with initial conditions corresponding to the unstable EO in Fig. 1, $R = 1$, $\varphi = 2.17$ and $\psi = 0.923$

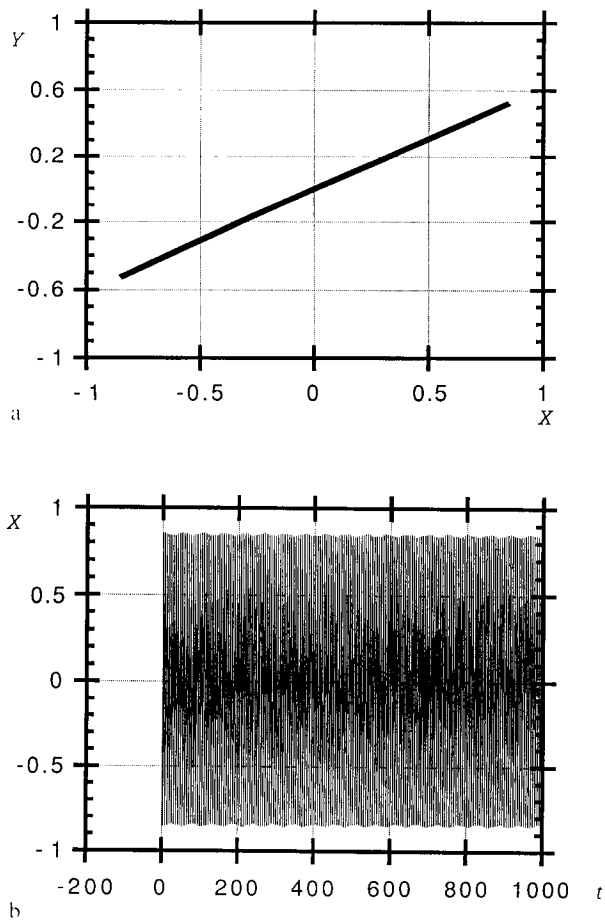


Fig. 3. Numerical integration of Eqs. (6), (7) for parameters (32) and $\varepsilon = 0.1$, with initial conditions corresponding to a stable NNM in Fig. 1, $R = 1$, $\varphi = 0$ and $\psi = 0.54$

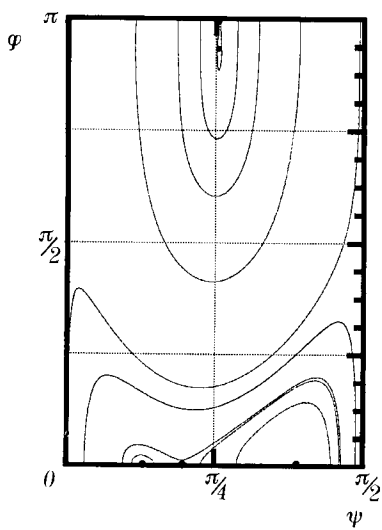
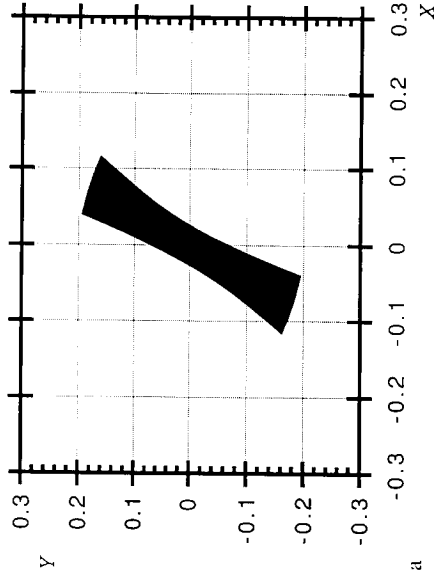
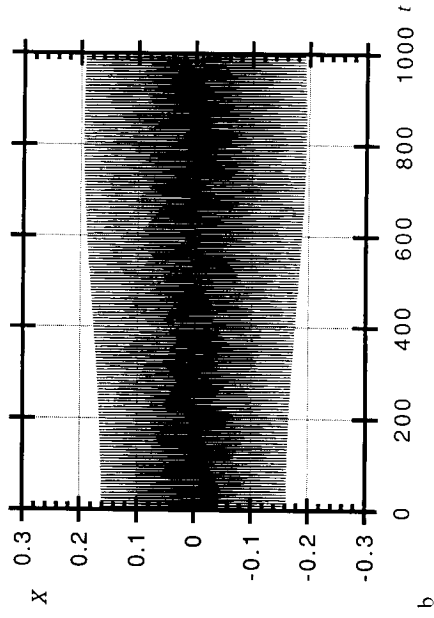


Fig. 4. Approximate first integral, Eq. (31), for parameters (41)

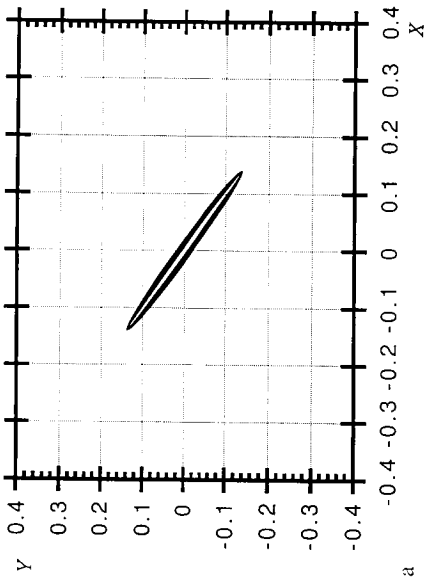


a

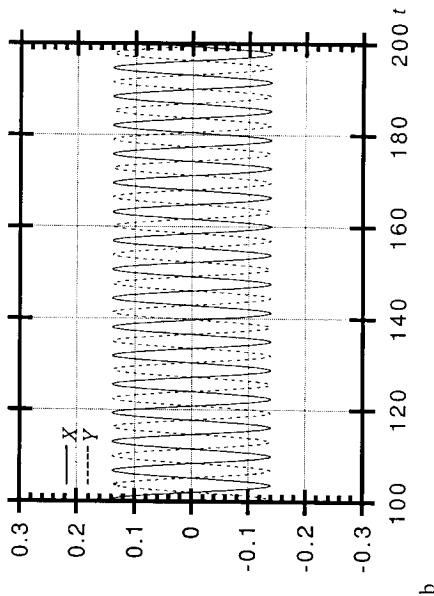


b

Fig. 6. Numerical integration of Eqs. (6), (7) for parameters (41) and $\varepsilon = 0.1$, with initial conditions corresponding to the unstable NNM in Fig. 4, $\varphi = 0$ and $\psi = 0.608$



a



b

Fig. 5. Numerical integration of Eqs. (6), (7) for parameters (41) and $\varepsilon = 0.1$, with initial conditions corresponding to the stable EO in Fig. 4, $\varphi = 2.88$ and $\psi = 0.799$

plane in Fig. 5 a and the phase differences between the coordinates $x(t)$ and $y(t)$ in Fig. 5 b. The NNM at $\varphi = 0$, $\psi = 0.608$ is examined in Fig. 6 and is shown to be orbitally unstable, in agreement with the analytical results. Note in this case the large amplitude modulations in the response, an indication of orbital instability.

5 Bifurcations

In the remainder of this paper we shall be interested in the number and type of PM's which occur in our system as a function of the parameter values.

As we move from point to point in the six dimensional parameter space of our system, the singular points of Eq. (28) (which correspond to PM's in the original system) and of its first integral (31) change their position and in exceptional cases change their number and type by merging with other singular points. Although the complete characterization of all such bifurcations in a six dimensional parameter space is too complicated a task, we are interested in characterizing the "worst case", i.e. the most degenerate singularity which may occur. This section contains an analysis of this problem. In the next section we seek a universal unfolding of the degenerate singularity.

The singular points of Eq. (28) are given by simultaneously requiring Eqs. (29) and (30) to be satisfied:

$$f(\varphi, \psi) = 0, \quad g(\varphi, \psi) = 0. \quad (42)$$

From Eq. (30), $g(\varphi, \psi) = 0$ may be satisfied either by taking

$$\sin \varphi = 0 \quad (43)$$

or by taking

$$\cos \varphi = \frac{3(a_{13} - a_{31}) \cos^2 \psi - 3a_{13}}{4a_{22} \cos \psi \sin \psi}. \quad (44)$$

When Eq. (43) is substituted into Eq. (29) for $f(\varphi, \psi) = 0$, we obtain conditions for NNM's (since these occur for $\varphi = 0$ or π). Similarly when Eq. (44) is substituted into $f(\varphi, \psi) = 0$, we obtain conditions for EO's. We will investigate these two cases separately, and we seek conditions on the parameters of the system such that the maximal degeneracy occurs in each. By simultaneously requiring both degeneracy conditions to hold, we will achieve the most degenerate singularity involving the bifurcation of both NNM's and EO's.

We begin by satisfying Eq. (41) with $\varphi = 0$ ($\varphi = \pi$ leads to the same results), which when substituted into Eq. (29) gives the following condition for NNM's:

$$-\frac{4}{3} \frac{\Delta}{R^2} p(p^2 + 1) + a_{13} p^4 + (2a_{22} - 4a_{04}) p^3 + (3a_{31} - 3a_{13}) p^2 + (4a_{40} - 2a_{22}) p - a_{31} = 0 \quad (45)$$

where

$$p = \tan \psi.$$

Since Eq. (45) is a quartic on p , the most degenerate case would correspond to a quadruple root, if the parameters could be chosen so that this were possible. This cannot happen however, as may be shown as follows. Let p_0 be the quadruple root. Then

$$(p - p_0)^4 = 0. \quad (46)$$

Comparing coefficients of like powers of p in Eq. (46) and Eq. (45)/ a_{13} , we obtain for the powers of p^0 and p^2 , respectively:

$$p_0^4 = -\frac{a_{31}}{a_{13}}, \quad 6p_0^2 = -3 + 3\frac{a_{31}}{a_{13}}. \quad (47)$$

Eliminating a_{31}/a_{13} in Eq. (47) shows that p_0 must satisfy the eq.

$$p_0^4 + 2p_0 + 1 = 0 \quad (48)$$

which has no real roots.

Since a quadruple root is impossible, we next consider that the maximal degeneracy in Eq. (45) could correspond to a triple root, which turns out to be possible. Let p_0 be the triple root, and let p_1 be the remaining root. Then

$$(p - p_0)^3 (p - p_1) = 0. \quad (49)$$

Comparing coefficients of like powers of p in Eq. (49) and Eq. (45)/ a_{13} , we obtain four equations from which we can eliminate p_0 and p_1 , giving two conditions on the parameters of the problem. Let us call these two resulting conditions Eq. (I) and Eq. (II). Inspection shows that Eq. (I) can be factored into what we shall call Eq. (Ia) and Eq. (Ib):

$$16 \left[\frac{\Delta}{R^2} \right]^2 - 48(a_{22} - 2a_{04}) \frac{\Delta}{R^2} + 9a_{31}^2 - 54a_{13}a_{31} + 36a_{22}^2 - 144a_{04}a_{22} + 81a_{13}^2 + 144a_{04}^2 = 0, \quad (Ia)$$

$$16a_{31} \left[\frac{\Delta}{R^2} \right]^2 - 48a_{31}(a_{22} - 2a_{04}) \frac{\Delta}{R^2} - 9a_{13}^2 + 54a_{13}a_{31} + 36a_{22}^2a_{31} - 144a_{31}a_{04}a_{22} - 81a_{13}a_{31}^2 + 144a_{04}^2a_{31} = 0, \quad (Ib)$$

$$314928a_{13}^2a_{22}^2a_{40} \frac{\Delta}{R^2} + 117 \text{ similar terms} = 0. \quad (II)$$

Equation (II) is a homogeneous polynomial of degree 6 in the parameters Δ/R^2 , a_{40} , a_{31} , a_{22} , a_{13} , a_{04} having 118 terms and is therefore too long to give here. Our conclusion so far is that in order for a system to exhibit a triply degenerate NNM, its parameters must satisfy either Eq. (Ia) or (Ib), and Eq. (II).

We continue by looking for the comparable maximally degenerate EO by satisfying $g(\varphi, \psi) = 0$ with Eq. (44). Substitution of Eq. (44) into Eq. (29) for $f(\varphi, \psi) = 0$ and solving for $\cos^2 \psi$ gives:

$$\cos^2 \psi = \frac{8a_{22}(\Delta/R^2) + 9a_{13}a_{31} - 4a_{22}^2 + 24a_{04}a_{22} - 9a_{13}^2}{-9(a_{13} - a_{31})^2 - 8a_{22}(a_{22} - 3a_{04} - 3a_{40})} \equiv Z. \quad (50)$$

If we represent the expression for $\cos^2 \psi$ in Eq. (50) by Z , Eq. (44) becomes

$$\cos^2 \varphi = \frac{9[(a_{31} - a_{13}) Z + a_{13}]^2}{16a_{22}^2(1 - Z) Z}. \quad (51)$$

Since $\cos^2 \varphi \leq 1$, the most degenerate case of Eq. (51) corresponds to $\cos^2 \varphi = 1$, which gives a quadratic on Z :

$$[9(a_{31} - a_{13})^2 + 16a_{22}^2] Z^2 + [18a_{13}(a_{31} - a_{13}) - 16a_{22}^2] Z + 9a_{13}^2 = 0. \quad (52)$$

The most degenerate case of Eq. (52) occurs when Z has a double root, which requires that the discriminant of (52) vanish, giving the condition:

$$a_{13} = \frac{4a_{22}^2}{9a_{31}} \quad (III)$$

in which case the double root Z becomes:

$$Z = \frac{4a_{22}^2}{9a_{31}^2 + 4a_{22}^2} \quad (53)$$

where Eq. (III) has been used to simplify the expression for Z . Note that $Z = \cos^2 \psi$ lies in between 0 and 1. Substituting Eq. (53) into Eq. (50) and using Eq. (III) to simplify the result gives:

$$\frac{\Delta}{R^2} = \frac{24a_{22}^2 a_{40} - 9a_{22} a_{31}^2 - 54a_{04} a_{31}^2 + 4a_{22}^3}{2(9a_{31}^2 + 4a_{22}^2)}. \quad (IV)$$

Our conclusion is that in order that a system exhibit a maximally degenerate EO, its parameters must satisfy Eqs. (III) and (IV).

So in order to obtain those systems which simultaneously exhibit maximal degeneracies for both NNM's and EO's, we must find those points in the six dimensional parameter space Δ/R^2 , a_{40} , a_{31} , a_{22} , a_{13} , a_{04} which satisfy the bifurcation Eqs. (II), (III), (IV) and either (I a) or (I b). Each of these equations may be written as homogeneous polynomials in these parameters. This permits us to normalize the parameters, thereby reducing the number from 6 to 5 without loss of generality. We do this by dividing each parameter by a_{31} , i.e. by defining new parameters as:

$$\hat{a}_{ij} = \frac{a_{ij}}{a_{31}}, \quad \hat{\Delta} = \frac{\Delta}{a_{31}}. \quad (54)$$

The resulting bifurcation equations will be the same as the ones we already have if we set

$$a_{31} = 1 \quad (55)$$

everywhere and replace the original parameters with the new ones. We do this, but for convenience we omit the $\hat{\quad}$ notation. So now we need only deal with 5 parameters, but the bifurcation eqs. are no longer homogeneous in these parameters.

The number of parameters that we need to consider can be further reduced by using Eqs. (III) and (IV) to respectively eliminate a_{13} and Δ/R^2 from Eqs. (I a), (I b) and (II). We shall refer to the resulting eqs. as Eqs. (I a*), (I b*) and (II*) respectively. These are greatly simplified, and depend only on a_{22} , a_{40} and a_{04} . Inspection of the resulting expressions shows that all occurrences of a_{04} and a_{40} are in the form $a_{04} + a_{40}$, which for convenience we shall refer to as Q :

$$Q = a_{40} + a_{04}. \quad (56)$$

Moreover, Eq. (I b*) is able to be factored into two simpler equations, which we shall refer to as Eqs. (I b1*) and (I b2*). Here is the form of the resulting bifurcation equations:

$$256a_{22}^8 + 1024a_{22}^6 - 1536a_{22}^5Q + 2304a_{22}^4Q^2 + 2016a_{22}^4 - 6912a_{22}^3Q + 3888a_{22}^2 + 729 = 0, \quad (\text{I a}^*)$$

$$16a_{22}^4 - 144a_{22}^2 + 216a_{22}Q - 567 = 0, \quad (\text{I b1}^*)$$

$$16a_{22}^4 - 216a_{22}Q + 81 = 0, \quad (\text{I b2}^*)$$

$$[16a_{22}^4 - 216a_{22}Q + 81] [55296a_{22}^8Q + 25 \text{ similar terms}] = 0. \quad (\text{II}^*)$$

Note that Eq. (II*) factors into Eq. (I b2*) as well as a long polynomial. This means that any points (a_{22}, Q) which satisfy Eq. (I b2*) also satisfy Eq. (II*), and hence are points of maximum degeneracy. Such points lie on the following curve in the $Q - a_{22}$ plane, see Fig. 7:

$$Q = \frac{2a_{22}^3}{27} + \frac{3}{8a_{22}}. \quad (57)$$

Have we found all such points (a_{22}, Q) which exhibit maximum degeneracy? In order to find out, we must see if Eq. (II*) can be solved simultaneously with either of Eqs. (I a*) or (I b1*). We first eliminate Q between Eqs. (II*) and (I a*). a step which can be conveniently accomplished in MACSYMA using the ELIMINATE command, which results in a polynomial on a_{22} which turns out to have no real roots. In the case of Eq. (I b1*), however, additional degenerate points do exist. This may be seen by eliminating Q between Eqs. (II*) and (I b1*), which gives the following additional points of maximum degeneracy, see Fig. 7:

$$(a_{22}, Q) = \left\{ \left(\frac{3}{2}, \frac{5}{2} \right), \left(\frac{-3}{2}, \frac{-5}{2} \right) \right\}. \quad (58)$$

The parameter values of a_{13} and Δ/R^2 which correspond to the degenerate points of Eqs. (57), (58) are obtained by substituting into Eqs. (III), (IV).

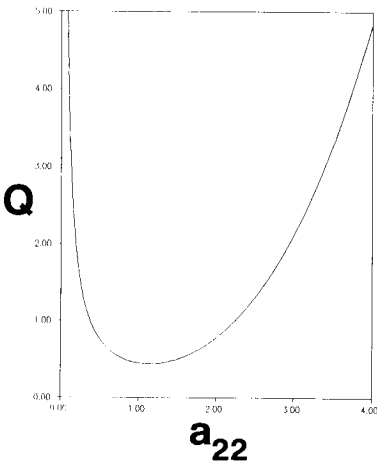


Fig. 7. Points of maximum degeneracy in $Q - a_{22}$ parameter space, see Eqs. (57) and (58). $Q = a_{40} + a_{04}$, Eq. (56)

6 Unfolding the singularity

We have shown that (to within the order of approximation of the perturbation method) there exist systems which exhibit maximally degenerate singularities. We now investigate systems which are close to these degenerate systems. The procedure involves prescribing a small but arbitrary change in parameters away from a degenerate point, in order to obtain all possible local bifurcations of PM's. The result is called a universal unfolding of the degenerate point [3].

Let a typical point on the curve (57) be parameterized by a_{22}^0 . Then the corresponding value of $Q = a_{40}^0 + a_{04}^0$ is given by Eq. (57), while the values of a_{13}^0 and Δ^0/R^2 are given by Eqs. (III) and (IV). The value of a_{31}^0 remains normalized at unity. Corresponding to these parameters, there is a degenerate singularity in the $\varphi - \psi$ plane located at

$$\varphi^0 = \pi \quad \text{and} \quad \psi^0 = \arctan \left[\frac{3}{2a_{22}^0} \right] \quad (59)$$

(respectively from Eqs. (43) and (53)). (We omit the choice $\varphi^0 = 0, \psi^0 = -\arctan(3/[2a_{22}^0])$ since it gives an equivalent unfolding to that of Eqs. (59).) Thus we set:

$$a_{22} = a_{22}^0 + b_{22}, \quad a_{04} = a_{04}^0 + b_{04}, \quad a_{40} = a_{40}^0 + b_{40}, \quad (60)$$

$$a_{13} = a_{13}^0 + b_{13}, \quad \frac{\Delta}{R^2} = \frac{\Delta^0}{R^2} + b_{\Delta}, \quad \varphi = \varphi^0 + u, \quad \psi = \psi^0 + v$$

where the b_{ij} 's are unfolding parameters (assumed small) and where u and v represent the associated displacement of the singular point from (φ^0, ψ^0) . Our goal is to obtain expressions for u and v in terms of the b_{ij} 's.

We begin by substituting Eqs. (59), (60) into Eqs. (29), (30) for $f(\varphi, \psi)$ and $g(\varphi, \psi)$. This gives two Eqs. on u and v with the solution $u = v = 0$ when all the $b_{ij} = 0$. We separately investigate NNM's and EO's. The eq. that comes from $g(\varphi, \psi) = 0$ is satisfied by $u = 0$, corresponding to NNM's (which satisfy $\varphi \equiv \varphi^0$). Substituting $u = 0$ into $f(\varphi, \psi) = 0$, Eq. (29), and Taylor-expanding the result for small v gives:

$$K_0 + K_1v + K_2v^2 + K_3v^3 + \dots = 0 \quad (61)$$

where the K_i are known functions of the b_{ij} 's as well as the a_{ij}^0 's. Since $f(\varphi, \psi)$ is linear in the a_{ij} 's, the K_i 's are affine in the b_{ij} 's.

In order to similarly treat the EO's, we solve $g(\varphi, \psi) = 0$ for $\cos \varphi$ as in Eq. (44), substitute the result into $f(\varphi, \psi) = 0$, and obtain Eq. (50) for $\cos^2 \psi$. Substituting Eqs. (59) and (60) into Eq. (50) and Taylor-expanding for small v gives:

$$L_0 + L_1v + \dots = 0 \quad (62)$$

where the L_i are known functions of the b_{ij} 's as well as the a_{ij}^0 's.

Conditions (58) on NNM's and (59) on EO's can be simplified without loss of generality by choosing b_{Δ} to make K_0 vanish and b_{13} to make L_0 vanish. Generality is assured because the results of the analysis will depend upon only two unfolding parameters (a "codimension 2"

bifurcation). We obtain:

$$b_A = \frac{3(32b_{40}a_{22}^{04} - 12b_{22}a_{22}^{02} - 72b_{04}a_{22}^{02} + 27b_{22})}{8a_{22}^0(4a_{22}^0 + 9)}, \tag{63}$$

$$b_{13} = \frac{b_{22}(4a_{22}^0 - 9)}{9a_{22}^0}. \tag{64}$$

With $K_0 = 0$, Eq. (58) has the root $v = 0$, as well as two additional possible roots which, if they exist, may be written in the approximate form

$$\text{NNM's: } u = 0, \quad v^2 \sim \frac{27}{a_{22}^0[4a_{22}^0 + 9]^4} \times [(64a_{22}^0 + 144a_{22}^0 + 540a_{22}^0 + 243) b_{22} - 576a_{22}^0(b_{04} + b_{40})]. \tag{65}$$

Eq. (62) with $L_0 = 0$ gives $v = 0$ for EO's. Substitution of this result and Eqs. (59), (60) into Eq. (51) and Taylor-expanding for small u gives an approximate expression for u in terms of the b_{ij} 's for EO's:

$$\text{EO's: } v = 0, \quad u^2 \sim \frac{4a_{22}^0 + 9}{4a_{22}^0} b_{22}. \tag{66}$$

Inspection of Eqs. (65), (66) shows that the unfolding parameters are naturally chosen as b_{22} and $(b_{04} + b_{40})$. Eq. (65) shows that in addition to the NNM at $u = v = 0$ (the degenerate singularity), two additional NNM's occur if

$$b_{22} > \frac{576a_{22}^0}{64a_{22}^0 + 144a_{22}^0 + 540a_{22}^0 + 243} (b_{04} + b_{40}). \tag{67}$$

Similarly, Eq. (66) shows that an EO (and it's mirror image in the line $\varphi = \pi$) occur if

$$b_{22} > 0. \tag{68}$$

The straight lines (67) and (68) are bifurcation curves in the unfolding, see Fig. 8. Another bifurcation curve is obtained by considering the stability of the EO. This may be ascertained as follows: Let the first integral (31) be called $h(\varphi, \psi) = \text{constant}$. Then at the EO singularity (66), h_φ and h_ψ vanish. The transition from stable to unstable occurs when the determinant of the

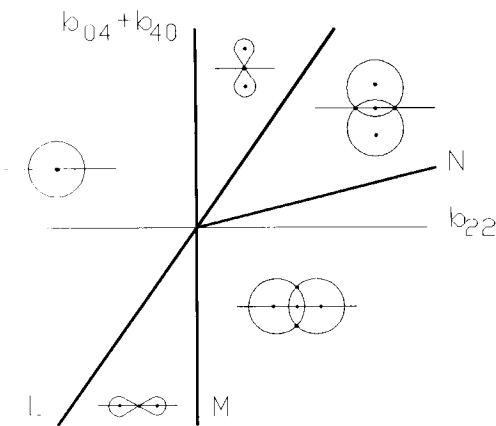


Fig. 8. Universal unfolding of a point of maximum degeneracy. The lines L , M and N respectively correspond to the conditions (67), (68) and (70). A phase portrait in the $\varphi - \psi$ plane is displayed in each region. The horizontal line in each phase portrait corresponds to $\varphi = \pi$. The phase portrait on line N consists of a circle of nonisolated singular points

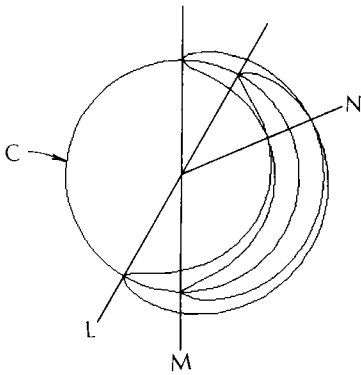


Fig. 9. Bifurcation diagram showing the location of singular points (PM's) which lie in the neighborhood of a point of maximum degeneracy. The circle *C* represents the maximally degenerate singularity, and the other curves represent singularities which have bifurcated from it as one moves in a circle around the origin in Fig. 8. The radial distance from the circle *C* represents the distance in the $\varphi - \psi$ phase plane from the maximally degenerate singularity to the bifurcated singularities. As in Fig. 8, the lines *L*, *M* and *N* respectively correspond to the conditions (67), (68) and (70)

Hessian vanishes:

$$\det \begin{bmatrix} h_{\varphi\varphi} & h_{\varphi\psi} \\ h_{\psi\varphi} & h_{\psi\psi} \end{bmatrix} = 0. \tag{69}$$

Substitution of Eqs. (59), (60) and (66) into (69) gives

$$b_{22} = \left[\frac{216a_{22}^{02}}{16a_{22}^{04} + 81} \right] (b_{04} + b_{40}). \tag{70}$$

Figure 8 shows the unfolding of a maximally degenerate singularity. The phase portraits displayed in Fig. 8 show the region of the $\varphi - \psi$ plane surrounding the degenerate singularity at $\varphi^0 = \pi$, and include a portion of the plane for which $\varphi > \pi$, which as we have noted previously, is a mirror image of $\varphi < \pi$. The phase portrait on the curve (70) involves a circle of nonisolated singular points.

If one were to traverse a circle around the origin in Fig. 8, the bifurcation diagram of Fig. 9 would be encountered. This may be described as two double pitchforks joined together by two transcritical bifurcations [3].

7 Conclusions

Previous studies of NNM's have shown how the number of NNM's may change through bifurcation [4, 10]. The present paper goes beyond previous work in several ways. To begin with, through the power of computer algebra, we have been able to study the general class of systems (1)–(7), rather than a specific system (cf. [4, 10]). For this class of systems, we have been able to obtain an approximate first integral which has permitted us to determine the number and stability of PM's. Moreover, for systems in the general class (1)–(7), we have been able to obtain the most degenerate possible bifurcation of PM's, and have obtained a universal unfolding which describes all possible local behavior of PM's. In particular, we have shown how NNM's are related to a family of PM's which we have called EO's. While NNM's project onto the $x - y$ configuration plane as line segments, EO's project as ellipses.

These results are seen as a first step in understanding the relationship between NNM's and more general classes of periodic orbits. It is expected that additional bifurcations exist in the class of systems considered here, and that approximate descriptions of such bifurcations may be obtained by extending the perturbation analysis to include higher order terms.

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References

- [1] Bauer, L., Reiss, E. L., Keller, H. B.: Axisymmetric buckling of hollow spheres and hemispheres. *Comm. Pure Applied Math.* **23**, 529–568 (1970).
- [2] Deprit, A., Henrard, J.: A manifold of periodic orbits. *Advances in Astronomy and Astrophysics* **6**, 1–124 (1968).
- [3] Guckenheimer, J., Holmes, P. J.: *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*. Berlin, Heidelberg, New York, Tokyo: Springer 1983.
- [4] Johnson, T. L., Rand, R. H.: On the existence and bifurcation of minimal normal modes. *Int. J. Nonlinear Mechanics* **14**, 1–12 (1979).
- [5] Kauderer, H.: *Nichtlineare Mechanik*. Berlin, Göttingen, Heidelberg, New York: Springer 1958.
- [6] Kevorkian, J., Cole, J. D.: *Perturbation methods in applied mathematics*. Berlin, Heidelberg, New York: Springer 1981.
- [7] Kinney, W., Rosenberg, R. M.: On steady state vibrations of nonlinear systems with many degrees of freedom. *J. Appl. Mech.* **33**, 406–412 (1966).
- [8] Manevich, L., Mikhlin, I.: On periodic solutions close to rectilinear normal modes. *PMM* **36**, 1051–1058 (1972).
- [9] Mikhlin, I.: Resonance modes of near-conservative systems. *PMM* **38**, 459–464 (1974).
- [10] Month, L. A., Rand, R. H.: An application of the Poincaré map to the stability of nonlinear normal modes. *J. Appl. Mech.* **47**, 645–651 (1980).
- [11] Nayfeh, A. H.: *Perturbation methods*. New York: Wiley 1973.
- [12] Pak, C. H.: On the stability behavior of bifurcated normal modes in coupled nonlinear systems. *J. Appl. Mech.* **56**, 155–161 (1989).
- [13] Pak, C. H., Rosenberg, R. M.: On the existence of normal mode vibrations in nonlinear systems. *Q. Applied Math.* **26**, 403–416 (1968).
- [14] Rand, R. H.: A higher order approximation for nonlinear normal modes in two degree of freedom systems. *Int. J. Nonlinear Mechanics* **6**, 545–547 (1971).
- [15] Rand, R. H., Armbruster, D.: *Perturbation methods, bifurcation theory and computer algebra*. Berlin, Heidelberg, New York, Tokyo: Springer 1987.
- [16] Rosenberg, R. M.: On nonlinear vibrations of systems with many degrees of freedom. *Advances in Applied Mechanics* **9**, 155–242 (1966).
- [17] Shaw, S. W., Pierre, C.: Normal modes and superposition in nonlinear oscillatory systems. Preprint (1991).
- [18] Szebehely, V.: *Theory of orbits*. New York: Academic Press 1967.
- [19] Vakakis, A. F.: Analysis and identification of linear and nonlinear modes in vibrating systems. Ph. D. Thesis, California Institute of Technology, Pasadena, California 1990.
- [20] Vakakis, A. F., Rand, R. H.: Normal modes and global dynamics of a two degree of freedom system. Submitted for publication to *Int. J. Nonlinear Mechanics* (1991).

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