

GEOMETRICAL DYNAMICS: A NEW APPROACH TO PERIODIC ORBITS AROUND L_4

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Abstract. Geometrical dynamics is the study of the geometry of the orbits in configuration space of a dynamical system without reference to the system's motion in time.

Generalized coordinates for the circular restricted problem of three bodies are taken as polar coordinates r, θ centered at the triangular libration point L_4 . A time-independent nonlinear second order ordinary differential equation for r as a function of θ is derived. Approximations to periodic solutions are obtained by perturbations and Fourier series.

1. Introduction

Geometrical dynamics is the study of the geometry of the orbits in configuration space of a dynamical system without reference to the system's motion in time.

For holonomic, scleronomous, conservative systems, a set of time-independent equations which describe the orbit may be obtained from Jacobi's form of the principle of least action (Goldstein, 1950, p. 232). For such systems having two degrees of freedom a single second order ordinary differential equation results. It has been independently derived by several authors (Whittaker, 1964, p. 389; Kauderer, 1958, p. 599; Knothe, 1964, p. 65) and has been the subject of several investigations in the area of nonlinear normal mode vibrations (Rosenberg, 1964; Rosenberg and Kuo, 1964; Rand, 1971).

If the dynamical system has two degrees of freedom but is rheonomous or is non-conservative then the aforementioned equation no longer holds. A method for deriving a similar equation for nonconservative systems has been presented (Jones, 1970).

The circular restricted problem of three bodies is rheonomous but Hamiltonian conserving (Goldstein, 1950, p. 55fn.). The kinetic energy in a coordinate system rotating with the primaries is not a homogeneous quadratic function of the generalized velocities (Goldstein, 1950, p. 23) and therefore Jacobi's form of the principle of least action is inapplicable (Goldstein, 1950, p. 232). Nonetheless the system has two degrees of freedom and a single time-independent second order ordinary differential equation describing the orbit of the third body can be derived by using the conservation of the Hamiltonian (Whittaker, 1964, p. 64). This equation has been presented for generalized coordinates taken as cartesian coordinates rotating with the primaries (Szebehely, 1967, p. 48).

In this work generalized coordinates for the circular restricted problem of three

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bodies are taken as polar coordinates r, θ centered at the triangular libration point L_4 . A time-independent nonlinear second order ordinary differential equation for r as a function of θ is derived. Approximations to periodic solutions are obtained by perturbations and Fourier series. These solutions represent periodic orbits around L_4 .

Periodic orbits around L_4 have been the subject of other investigations (Szebehely, 1967; Pederson, 1935; Deprit and Delie, 1965) based on the usual dynamical approach in which time appears as an independent variable. It is the purpose of this work to suggest, through geometrical dynamics, an alternative approach.

2. Derivation of the Geometrical Dynamics Equation

The equations of motion for the circular restricted problem of three bodies in the usual dimensionless barycentric synodic coordinates x, y are (Szebehely, 1967, p.18)

$$\begin{aligned} \ddot{x} - 2\dot{y} - x &= -V_x \\ \ddot{y} + 2\dot{x} - y &= -V_y \end{aligned} \tag{1}$$

where

$$V = - (1 - \mu)/r_1 - \mu/r_2 \tag{2}$$

$$\begin{aligned} r_1^2 &= (x - \mu)^2 + y^2 \\ r_2^2 &= (x + 1 - \mu)^2 + y^2. \end{aligned} \tag{3}$$

Here V is the gravitational potential energy of the primaries and μ is the ratio of the mass of the smaller primary to the total mass of the primaries. The coordinate system rotates in a Newtonian inertial frame with dimensionless angular velocity unity.

In order to obtain the equations of motion in terms of polar coordinates r, θ centered at L_4 , set

$$\begin{aligned} x &= a + r \cos(\theta + \alpha) \\ y &= b + r \sin(\theta + \alpha) \end{aligned} \tag{4}$$

where

$$\begin{aligned} a &= \mu - 1/2 \\ b &= \sqrt{3}/2 \\ \tan 2\alpha &= \sqrt{3}(1 - 2\mu). \end{aligned} \tag{5}$$

Here (a, b) is the triangular libration point L_4 . Principal coordinates result from the rotation through angle α (Szebehely, 1967, p. 254).

Equations (1) become

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 - 2r\dot{\theta} &= -U_r \\ r^2\ddot{\theta} + 2r\dot{r}(\dot{\theta} + 1) &= -U_\theta \end{aligned} \tag{6}$$

where

$$U(r, \theta) = V - r^2/2 - r[a \cos(\theta + \alpha) + b \sin(\theta + \alpha)]. \tag{7}$$

Equations (6) admit the first integral

$$\dot{r}^2/2 + r^2\dot{\theta}^2/2 + U = h \quad (8)$$

where h is a constant.

Now consider r as a function of θ only.

Then

$$\begin{aligned} \dot{r} &= r'\dot{\theta} \\ \ddot{r} &= r''\dot{\theta}^2 + r'\ddot{\theta} \end{aligned} \quad (9)$$

where primes denote differentiation with respect to θ . Solving (6) for \ddot{r} and $\ddot{\theta}$, replacing \dot{r} wherever it appears by $r'\dot{\theta}$, solving (8) for $\dot{\theta}$ and finally substituting all of these results into the second of (9) gives

$$\begin{aligned} 2(h - U)(-rr'' + r^2 + 2r'^2) - 2[2(h - U)]^{1/2}[r^2 + r'^2]^{3/2} + \\ + (r^2 + r'^2)(-rU_r + (r'/r)U_\theta) = 0. \end{aligned} \quad (10)$$

Here it has been assumed that $\dot{\theta} < 0$ for periodic orbits around L_4 (Szebehely, 1967; Pederson, 1935; Deprit and Delie, 1965).

3. An Approximate Solution

Expanding $U(r, \theta)$ in a power series in r ,

$$U = -1 + r^2g(\theta) + r^3f(\theta) + 0(r^4) \quad (11)$$

where

$$g(\theta) = (3/4)(-1 + \lambda \cos 2\theta) \quad (12)$$

$$\begin{aligned} 16f(\theta) = 3\sqrt{3} \sin(\theta + \alpha) - 3(1 - 2\mu) \cos(\theta + \alpha) + \\ + 10(1 - 2\mu) \cos 3(\theta + \alpha) \end{aligned} \quad (13)$$

$$\lambda = [1 - 3\mu(1 - \mu)]^{1/2}. \quad (14)$$

It is well known that if terms of $O(r^3)$ are neglected in U , then there exists an elliptic solution to (10) of the form (Szebehely, 1967, p. 258)

$$r^2 = M/(N + \cos 2\theta) \quad (15)$$

where M and N are constants. Substituting (15) into (10) and neglecting $O(r^3)$ in (11), find

$$M = -16|h + 1|/3\lambda(9\lambda^2 - 8)^{1/2} \quad (16)$$

$$N = -[4 + (9\lambda^2 - 8)^{1/2} \operatorname{sgn}(h + 1)]/3\lambda. \quad (17)$$

Note that there is exactly one periodic orbit (15) for given values of h and μ . The value $h = -1$ corresponds to the equilibrium solution L_4 . Values of $h > -1$ (< -1) correspond to the so called short (long) period orbits (Szebehely, 1967, p. 258).

The condition that M and N be real,

$$9\lambda^2 - 8 > 0 \tag{18}$$

is equivalent to the usual stability criterion,

$$\mu(1 - \mu) < 1/27. \tag{19}$$

In order to find periodic solutions to (10) when higher order terms are included in U , note that, from (15)–(17), $r = O(|h + 1|^{1/2})$. This suggests setting

$$r(\theta) = [M/(N + \cos 2\theta)]^{1/2} + x_2(\theta) \varepsilon^2 + x_3(\theta) \varepsilon^3 + \dots \tag{20}$$

where

$$\varepsilon = |h + 1|^{1/2}. \tag{21}$$

Substituting the perturbation scheme of Equation (20) into (10) and (11) and equating to zero coefficients of like powers of ε , obtain a set of linearized differential equations on the $x_n(\theta)$. After some algebra, the equation on $x_2(\theta)$ becomes

$$F_1(\theta) x_2'' + F_2(\theta) x_2' + F_3(\theta) x_2 = F_4(\theta) \tag{22}$$

where

$$F_1(\theta) = \sum_{\substack{n=0 \\ n \text{ even}}}^{10} \alpha_{1n} \cos n\theta$$

$$F_2(\theta) = \sum_{\substack{n=2 \\ n \text{ even}}}^{10} \alpha_{2n} \sin n\theta$$

$$F_3(\theta) = \sum_{\substack{n=0 \\ n \text{ even}}}^{10} \alpha_{3n} \cos n\theta$$

$$F_4(\theta) = \sum_{\substack{n=1 \\ n \text{ odd}}}^7 \alpha_{4n} \cos n\theta + \alpha_{5n} \sin n\theta$$

where the α_{in} are known constants which depend on λ and N only.

For a periodic solution to (22), it is sufficient to set

$$x_2(\theta) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} a_n \cos n\theta + b_n \sin n\theta. \tag{23}$$

Substituting the Fourier series (23) into (22) and equating to zero coefficients of $\cos n\theta$ and $\sin n\theta$, obtain a set of linear algebraic equations on the a_n and b_n respectively.

Periodic expressions for $x_3(\theta)$ and higher order terms could be found in a similar fashion.

For given values of h and μ there exists exactly one periodic orbit of the form (20), an analytic continuation of the elementary orbit (15).

As an example, consider the periodic orbit for $\mu=0.012$ and $h < -1$ (long period). It is approximated by the expression

$$r(\theta) = [-6.586/(-1.078 + \cos 2\theta)]^{1/2} \varepsilon + (-0.191 \cos \theta + 14.888 \sin \theta - 0.034 \cos 3\theta + 2.687 \sin 3\theta + \dots) \varepsilon^2 + O(\varepsilon^3). \quad (24)$$

This orbit is shown in Figure 1 for $h = -1.0009$ ($\varepsilon = 0.03$).

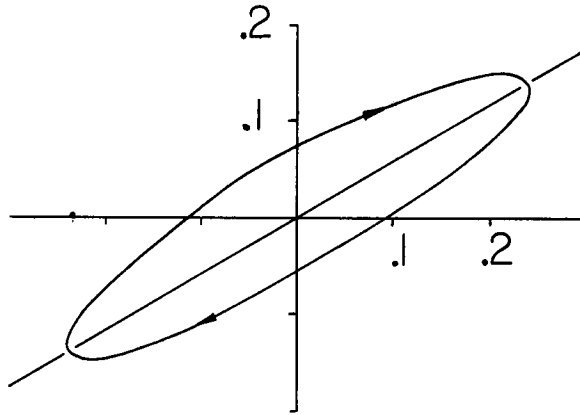


Fig. 1. Periodic orbit around L_4 for $\mu=0.012$ and $h = -1.0009$, Equation (24). The unit of length is the distance between the primaries. The horizontal line is parallel to the line connecting the primaries. The inclined line makes an angle α with the horizontal.

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