# On the Stability of a Differential Equation With Application to the Vibrations of a Particle in the Plane

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*The stability of the equation $\ddot{Z} + f(t)Z = 0$, where $f(t) = (\delta - \epsilon \cos^2 t)/(1 - \epsilon \cos^2 t)$, is studied by using Floquet theory, Fourier analysis and perturbations. The results are used to study the stability of the vibrations of a particle constrained to a plane and restrained by two identical linear springs with initial stress.*

THE DIFFERENTIAL equation

$$\ddot{Z} + f(t)Z = 0 \tag{1}$$

where

$$f(t) = (\delta - \epsilon \cos^2 t)/(1 - \epsilon \cos^2 t) \tag{2}$$

and where $\delta$ and $\epsilon$ are parameters and dots represent differentiation with respect to $t$, is a particular case of Hill's equation. For a given $\delta$ and $\epsilon$, the point $(\delta, \epsilon)$ is said to be stable if all solutions of equation (1) are bounded for all $t > 0$, and unstable if an unbounded solution exists. It is desired to find those regions in the $\delta - \epsilon$ plane which are stable.

Clearly if $\delta = 1$, $f(t) \equiv 1$ and (1) has only bounded solutions for all $\epsilon$. Therefore, the line $\delta = 1$ in the $\delta - \epsilon$ plane is stable. However, if $\epsilon \geq 1$ and $\delta \neq 1$, $f(t)$ becomes infinite when $\cos^2 t = 1/\epsilon$, and there will exist unbounded solutions to equation (1) for all $\delta \neq 1$. The half plane $\epsilon \geq 1$ is thus unstable except for the line $\delta = 1$ which is stable. In what follows, $\epsilon$ is restricted to values less than unity.

It is well known from Floquet theory (Stoker [1],[3] p. 201) that corresponding to transition values of $\delta$ and $\epsilon$ from stability to instability, there must exist at least one periodic solution to equation (1) of period $\Omega$ or $2\Omega$, where $\Omega$ is the period of $f(t)$. From equation (2), $\Omega = \pi$. Therefore, in order to obtain all transition values of $\delta$ and $\epsilon$, it is sufficient to examine solutions of period $2\pi/N$, all of which have period $2\pi$. (Here and in what follows, $M, N = 0, 1, 2, \ldots$)

Now for $\epsilon = 0$ and $\delta > 0$, the solutions to equation (1) are of the form $\sin \sqrt{\delta} t$ and $\cos \sqrt{\delta} t$, which have period $2\pi/\sqrt{\delta}$. Thus for $\epsilon = 0$, transition points can occur only if

$$2\pi/\sqrt{\delta} = 2\pi/N$$

or

$$\delta = N^2$$

Note that $N = 0$ corresponds to a constant, which is a solution to equation (1) when $\delta = \epsilon = 0$, and which may be thought of as a periodic function of period $2\pi$.

For $\epsilon = 0$ and $\delta \leq 0$, equation (1) has unbounded solutions, and hence the entire negative $\delta$-axis is unstable.

Thus, one expects two transition curves to intersect each of the foregoing transition points on the $\delta$-axis, one behaving like $\sin Nt$, the other like $\cos Nt$ for $\epsilon = 0$. (Except for $N = 0$, where one expects a single transition curve behaving like a constant for $\epsilon = 0$.) This is the situation for Mathieu's equation, for example (McLachlan [2], p. 40).

If a solution to equation (1) has period $2\pi$, it can be expanded in a Fourier series,

$$Z(t) = \sum_{N=0}^{\infty} (a_N \cos Nt + b_N \sin Nt) \tag{3}$$

Substituting equation (3) into equation (1) and equating coefficients of like terms results in four sets of infinite-order, homogeneous linear algebraic equations for the $a_N$ and $b_N$. Each set deals exclusively with $a_{2M}$, $b_{2M}$, $a_{2M+1}$, $b_{2M+1}$, respectively. Since each system of equations is homogeneous, the determinants of the coefficients must vanish for a nontrivial solution. This gives four infinite determinants, each representing a functional relation between $\delta$ and $\epsilon$, i.e., each representing a curve in the $\delta - \epsilon$ plane along which periodic solutions of period $2\pi$ exist. The curves so obtained from the $a_{2M}$ and $b_{2M}$ sets intersect the $\delta$-axis at $\delta = 4M^2$, while the curves obtained from the $a_{2M+1}$ and $b_{2M+1}$ sets intersect the $\delta$-axis at $\delta = (2M + 1)^2$.

Now since the coefficients of the $a_{2M}$ and $b_{2M}$ sets of equations are in general different, two different curves will intersect the $\delta$-axis at $\delta = 4M^2$, each of which will be a transition curve separating a region of stability from a region of instability.

However, the coefficients of the $a_{2M+1}$ and $b_{2M+1}$ sets of equations turn out to be exactly the same. Clearly, the same determinants will yield the same curves in the $\delta - \epsilon$ plane. There will be only one curve passing through the points $\epsilon = 0$, $\delta = (2M + 1)^2$, and on this curve there will exist two periodic solutions to equation (1) which are linearly independent. Since all other

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[3] Numbers in brackets designate References at end of Note.

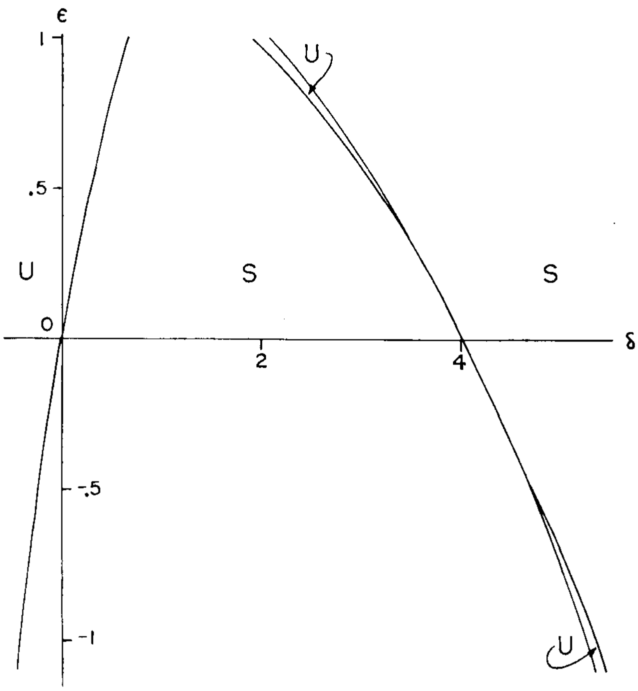


**Fig. 1 Stability diagram for equation (1). $U$ = unstable, $S$ = stable**

solutions for points on these curves can be expressed as a linear combination of these two solutions, the curves themselves are stable. They are not transition curves since the usual region of instability between the two curves has vanished. This situation is quite unlike Mathieu's equation; it is as if the two transition curves intersecting the $\delta$-axis at $\delta = (2M + 1)^2$ have coalesced!

To obtain explicit expressions for the transition curves which intersect the points $\epsilon = 0$, $\delta = 4M^2$, a perturbation method is used (Stoker [1], p. 209).

Expand

$$Z(t) = Z_0(t) + Z_1(t)\epsilon + Z_2(t)\epsilon^2 + \ldots \quad (4)$$

$$\delta = 4M^2 + \delta_1\epsilon + \delta_2\epsilon^2 + \ldots \quad (5)$$

and substitute equation (4) and (5) into equation (1). By equating the coefficients of like powers of $\epsilon$, obtain a linear differential equation with constant coefficients on $Z_N(t)$. Requiring $Z_N(t)$ to be periodic gives a value for $\delta_N$. For $M > 0$, $Z_0$ is taken first as $\cos 2Mt$, then as $\sin 2Mt$, since each choice gives a separate transition curve.

In this manner the equations for the first five transition curves to $0(\epsilon^5)$ were obtained:

$$\delta = (1/2)\epsilon + (3/32)\epsilon^2 + (3/64)\epsilon^3 + (975/32768)\epsilon^4 + 0(\epsilon^5) \quad (6)$$

$$\delta = 4 - (3/2)\epsilon - (21/64)\epsilon^2 - (21/128)\epsilon^3 - (13545/131072)\epsilon^4 + 0(\epsilon^5) \quad (7)$$

$$\delta = 4 - (3/2)\epsilon - (15/64)\epsilon^2 - (15/128)\epsilon^3 - (9705/131072)\epsilon^4 + 0(\epsilon^5) \quad (8)$$

$$\delta = 16 - (15/2)\epsilon - (45/32)\epsilon^2 - (45/64)\epsilon^3 - (29145/65536)\epsilon^4 + 0(\epsilon^5) \quad (9)$$

$$\delta = 16 - (15/2)\epsilon - (45/32)\epsilon^2 - (45/64)\epsilon^3 - (29115/65536)\epsilon^4 + 0(\epsilon^5) \quad (10)$$

The first three of these curves are shown in Fig. 1.

## Application

Consider a particle with coordinates $(x, y)$ constrained to the $x$-$y$ plane and restrained by two identical linear springs with initial stress, as in Fig. 2.

Choose the unit of mass such that the mass of the particle is unity, the unit of length such that the distance from the mass at rest to either support is unity, and the unit of time such that the sum of the two spring constants is unity. Let $L$ be the original length of the spring and let $\delta$ be the change in length of the spring when $x = y = 0$, relative to the original length. Then

$$\delta = 1 - L$$

E.g., $\delta > 0$ implies there is a tensile stress in both springs when $x = y = 0$. Clearly $\delta < 1$ since $L > 0$.

A possible mode of vibration, the $x$-mode, is

$$x = \sqrt{\epsilon} \cos t$$

$$y = 0$$

where $\sqrt{\epsilon}$ is a constant, the amplitude of vibration. Clearly $0 < \epsilon < 1$.

Yang and Rosenberg [3] have shown that the motion of a particle in the plane is governed by two coupled nonlinear ordinary differential equations, and that the stability of the $x$-mode is governed by two uncoupled linear ordinary differential equations, the equations of first variation. One of the first variational equations is stable for all values of $\epsilon$ and $\delta$, while the other is of the form of equation (1). Thus the stability of the $x$-mode is dependent only on the stability of equation (1).

By assuming that $\epsilon$ was small compared to unity, Yang and Rosenberg reduced equation (1) to Mathieu's equation, an approximation valid to $0(\epsilon^2)$. The results of their stability analysis, also valid to $0(\epsilon^2)$, are shown in Fig. 3, together with equation (6)

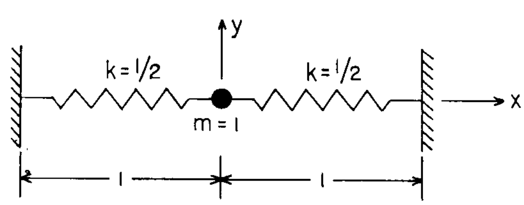


**Fig. 2 A particle in the plane**

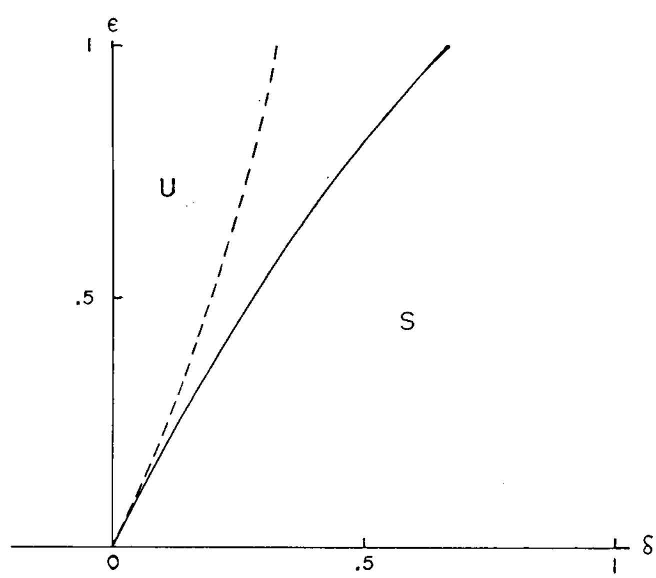


**Fig. 3 Stability of the x-mode of vibration of a particle in the plane. Dashed line represents the results of Yang and Rosenberg. Solid line represents equation (6) of this work. $U$ = unstable, $S$ = stable**

of this work, valid to $0(\epsilon^5)$. There is a significant difference in results for moderate values of $\epsilon$.

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## References

1 Stoker, J. J., *Nonlinear Vibrations*, Interscience, New York 1950.

2 McLachlan, N. W., *Theory and Application of Mathieu Functions*, Dover, N. Y., 1964.

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