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The Dynamics of Resonant Capture

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Abstract. Resonant capture describes the behavior of a weakly coupled multi-degree-of-freedom system when two or more of its uncoupled frequencies become locked in resonance. Flow on the region of phase space near the resonance (the resonance manifold) involves a region bounded by a separatrix in the uncoupled ($\varepsilon = 0$) system. Capture corresponds to motions which appear to cross into the interior of the separated region for $\varepsilon > 0$.
We offer two approximate methods for estimating which initial conditions lead to capture: an energy method and a perturbation method based on invariant manifold theory. These methods are applied to a model problem involving the spinup of an unbalanced rotor attached to an elastic support.

Key words: Resonance, capture, separatrix, invariant, manifold.

Introduction

The phenomenon of *resonant capture*, which has been the subject of a number of recent papers [2–12, 14–19, 21], may be described as follows. The setting is the phase space of a dynamical system which is assumed to contain one or more *resonance manifolds*. These are characterized by the failure in their neighborhood of averaging-type methods, i.e., a resonant manifold corresponds to a vanishing denominator in the averaged equations. A motion moving through the phase space is able to do so according to the averaged equations with little error except in the neighborhood of a resonant manifold, where the averaged equations no longer hold. Once such a motion enters the neighborhood of a resonant manifold, it undergoes more complicated local dynamics and it may, at some later time, *pass through* this neighborhood and continue on through the rest of the phase space according to the averaged equations, or it may remain in the neighborhood of the resonant manifold for all time, in which case it is said to be *captured*.

This problem may be studied by restricting interest to the dynamics of the resonant manifold itself. The flow on this subspace typically contains a *separatrix* which slowly evolves in time, thus permitting trajectories to pass across it. That is, the locally valid equations of motion involve a small parameter ε, which when taken equal to zero correspond to a dynamical system which has a genuine separatrix, but which does not contain a separatrix when $\varepsilon > 0$. Typically, the resonant manifold is two-dimensional, and the separatrix in the $\varepsilon = 0$ system encloses a region which contains a center. The $\varepsilon > 0$ dynamics may permit a motion to cross into the interior of the region defined by the separatrix, with the result that the motion begins to circulate around the center. If the motion enters the inside of the separated region it is said to be *captured*. In some problems a motion which has entered the separated region may leave it by again crossing the separatrix. Such a motion is said to have *escaped* capture. This term is also applied to motions which come close to the separated region but do not enter it. In some problems a typical motion enters and leaves the separated region many times, in some cases chaotically. It should be mentioned that in some problems the region we have called the resonant manifold, which contains the separatrix crossing problem, is the entire phase space.

Many researchers have approached the separatrix crossing problem by computing the energy change involved in a single passage around the separatrix loop [5–8, 11, 12, 14–16]. For small values of ε, the nearly-frozen separatrix will change its location and shape much more slowly than a motion circulating in its neighborhood. This permits many orbits of the circulating motion to occur before the separatrix is crossed. By knowing the energy change involved in a single such passage, it is possible to estimate the number of orbits before separatrix crossing. In problems where a motion makes a single pass in the neighborhood of the separatrix, either to be captured or to escape, the energy computation permits the initial condition leading to capture to be found.

In this paper we shall (i) generalize the energy approach presented previously by other researchers, and (ii) present a new approach to the problem of separatrix crossing and resonant capture. Before proceeding to the body of the paper, we offer the following description of (ii).

In the $\varepsilon = 0$ frozen system, we assume there is a saddle equilibrium together with a separatrix connecting its stable (S) and unstable (U) manifolds. For $\varepsilon > 0$, these features change their form. The saddle equilibrium is replaced by a hyperbolic motion (HM) which can be found by using invariant manifold theory. The separatrix connection is structurally unstable and will generally be broken by taking $\varepsilon > 0$. It is replaced by the S and U manifolds of the HM, which can be found by using a perturbation method once the HM has been found. The significance of the S and U manifolds of the HM is that they separate those motions which are capture from those which escape. By finding the location of these manifolds at time $t = 0$ we may obtain those initial conditions which lead to capture.

Model Problem

As a model problem which exhibits resonant capture we shall take the two degree of freedom system shown in Figure 1. This system consists of an unbalanced rotor attached to an elastic support and driven by a constant torque. It has been previously studied using other methods [17, 19, 21]. Typical behavior is shown in Figure 2. When the system is started from rest, the angular velocity of the rotor increases until it reaches the neighborhood of the natural frequency of the spring-mass system. Then, depending upon initial conditions, the rotor's angular velocity either continues to increase beyond the resonance region (pass-through), or it remains close to the natural frequency of the mass-spring system (capture). In the four-dimensional phase space, the resonance manifold is the hyperplane $d\theta/dt = 1$, where unity is the natural frequency of the spring-mass system.

The equations which govern the dynamics of the system in Figure 1 can be simplified by transforming coordinates and averaging, see Appendix A. The derivation is approximate, and assumes that the unbalance and applied torque are small and that the rotor motion is close to resonance. The result is a simplified system which is valid in the neighborhood of the resonance manifold:

$$\frac{d^2q}{dt^2} - q^2 + \omega\left[1 - \frac{1}{2}q^2\right] = 0 \tag{1}$$

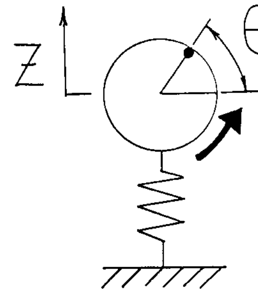

Fig. 1. A system consisting of an unbalanced rotor attached to an elastic support and driven by a constant torque [17, 19, 21]. See Appendix A.

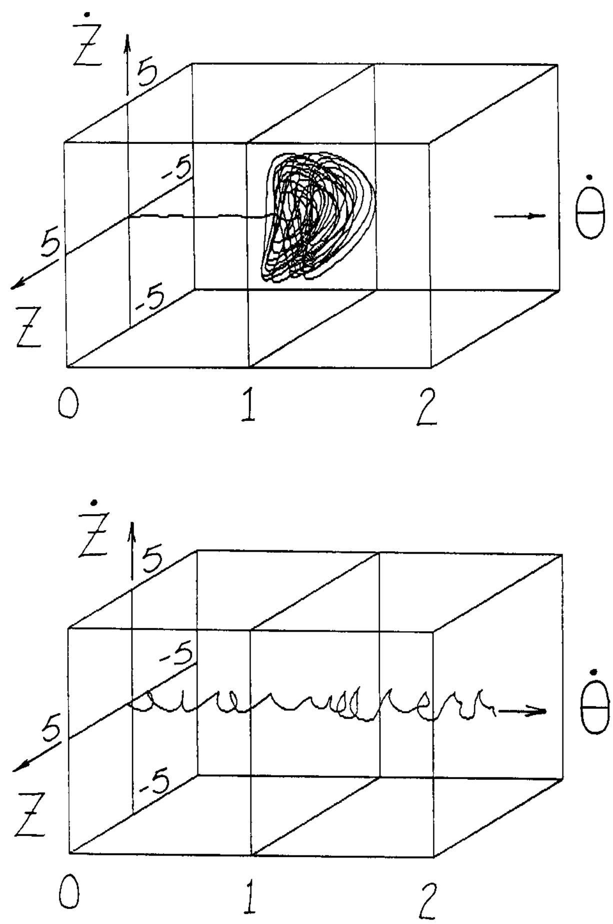

Fig. 2. Dynamics of system of Figure 1, equations (A1) and (A2). As the angular velocity $\dot{\theta}$ approaches the resonant manifold $\dot{\theta} = 1$, either the motion is 'captured' (top) or it 'passes through' the resonance (bottom). The question of which outcome occurs depends both on parameter values and initial conditions. Both cases correspond to $e = 0.1$, $K = 0.25$, $\theta(0) = \dot{\theta}(0) = \dot{z}(0) = 0$. The top case corresponds to $z(0) = 0$, while the bottom case corresponds to $z(0) = 0.4$.

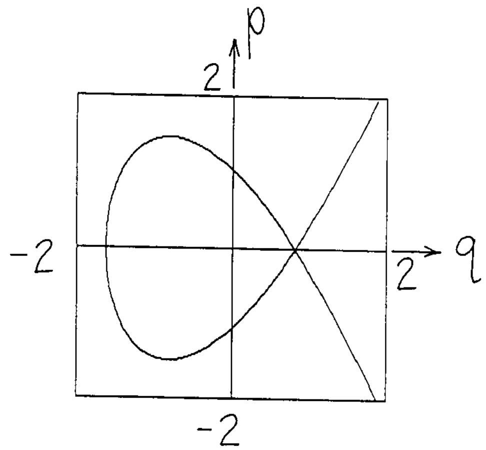

Fig. 3. Instantaneous phase portrait in the q–$\dot{q}$ phase plane for system (1), (2) for $\omega = 1$. Curves lying inside the separatrix loop are associated with capture. Curves lying outside of the separatrix loop are associated with pass-through.

$$\frac{d\omega}{dt} = \varepsilon \left[1 - \frac{1}{2} q^2\right], \quad \varepsilon \ll 1. \tag{2}$$

This system may be thought of as a strongly nonlinear oscillator q driven parametrically by a slowly changing feedback variable ω.

When $\varepsilon = 0$, equation (2) shows that ω is a constant. In this limiting case, there are equilibria at

$$q = \pm\sqrt{\frac{\omega}{1 + (\omega/2)}}, \quad \varepsilon = 0 \tag{3}$$

(assuming $\omega > 0$), and the q–$\dot{q}$ phase plane has a separatrix as shown in Figure 3. Motions which start inside (outside) the separatrix cannot cross it and so remain inside (outside) if for all time.

When $\varepsilon > 0$, equation (2) show that ω varies slowly in time. We may think about this case by imagining a series of 'instantaneous phase portraits', each as in Figure 3 for a constant value of ω, strung together in time as if ω was changing quasi-statically. Each such phase portrait will be said to contain 'instantaneous integral curves', one of which will be referred to as an 'instantaneous separatrix'. This notation, while natural and convenient, must be accompanied by a caveat: An instantaneous separatrix is not a true separatrix in the sense that a motion can cross it. In particular, a motion which starts on an instantaneous integral curve which lies outside the separatrix, may find itself inside the separatrix at some later time. It is this situation which characterizes resonant capture.

In order to observe the resonant capture phenomenon in equations (1) and (2), we used numerical integration and noted which initial conditions led to capture, see Figure 4. In contrast to these numerically obtained results, we shall, in the rest of the paper, be interested in developing analytical approximations for those initial conditions which lead to capture.

Energy Method

In order to determine when a motion will cross a separatrix loop, call it L, Henrard [11, 12], Cary *et al.* [8] and others have estimated the energy change for small ε involved in a

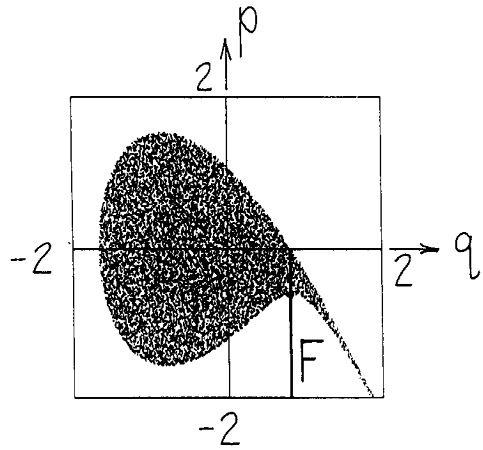

Fig. 4. Results of numerical integration of equation (1) for $\varepsilon = 0.1$ and $\omega(0) = 1$. Each dot represents an initial condition which leads to capture. Also displayed is a family F of initial conditions of the form $q(0) = \sqrt{2/3}$, $p(0) < 0$. The choice $p(0) = p^* = 0.585$ (shown as a large dot on F) corresponds to the critical initial condition separating those initial conditions in F which lead to capture from those which lead to escape.

single passage around L. These energies are referred to the instantaneous integral curves in the unperturbed conservative system. By knowing a motion's initial energy one may predict whether or not it will have crossed the instantaneous separatrix after a passage around L.

The previous results given in [5, 8, 11, 12] apply to a system which involves the slow forcing of an unperturbed system. The slowly varying forcing parameter ω has been assumed to increase linearly in time, i.e., $\dot{\omega} = \varepsilon$. In what follows we generalize the results in [5, 8, 11, 12] to permit ω to satisfy a more complicated evolution equation. In this section we summarize our results, the derivation of which is given in Appendix B. In the next section we will apply this approach to the model problem of equations (1) and (2).

Consider the forced 'one and a half' degree of freedom system:

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H}{\partial p}(q, p, \omega) + \varepsilon g_1(q, p, \omega) \tag{4}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial q}(q, p, \omega) + \varepsilon g_2(q, p, \omega) \tag{5}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \varepsilon f(q, p, \omega). \tag{6}$$

Here we assume $\varepsilon \ll 1$. Thus ω is a slowly varying parameter. When $\varepsilon = 0$, the system is conservative with Hamiltonian $H(q, p, \omega)$. We further assume that the $\varepsilon = 0$ system has a separatrix or 'saddle-loop', i.e. a homoclinic orbit or a pair of heteroclinic orbits. Our results consist of an estimate for the energy change Δh which occurs in the $\varepsilon > 0$ system, as a motion travels in the neighborhood of the separatrix. In this first-order approximation, the energy change Δh is referred to the $\varepsilon = 0$ phase portrait, i.e., the level sets of H.

We find that

$$\Delta h = \varepsilon \oint \left[g_1 \frac{\partial P}{\partial q} + g_2 - f \frac{\partial P}{\partial \omega} \right] \bigg|_{(h_{\mathrm{sep}}, \omega_0)} \mathrm{d}q, \tag{7}$$

where P is obtained by solving $H(p,q,\omega) = h$ for $p = P(q,\omega,h)$. The integral is taken along the instantaneous position of the separatrix in the q–p plane, holding the slowly varying quantity ω fixed at its initial value ω_0. This formula extends previous work by Henrard [11, 12], Carey *et al.* [8] and others, who studied the special case $g_1 = g_2 = 0$, f = constant. Equation (7) agrees with the formula of Melnikov which is usually used to predict the intersection of stable and unstable manifolds [15].

Application of Energy Method to Model Problem

We write equations (1) and (2) in the form:

$$\frac{\mathrm{d}q}{\mathrm{d}t} = p \tag{8}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = q^2 - \omega\left[1 - \frac{1}{2}q^2\right] \tag{9}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \varepsilon\left[1 - \frac{1}{2}q^2\right]. \tag{10}$$

This system has an unperturbed Hamiltonian given by:

$$H(q,p,\omega) = \frac{p^2}{2} - \frac{q^3}{3} + \omega\left[q - \frac{q^3}{6}\right]. \tag{11}$$

Comparison with equations (4)–(6) gives:

$$g_1 = 0, \quad g_2 = 0, \quad f = \left[1 - \frac{1}{2}q^2\right]. \tag{12}$$

Solving the Hamiltonian for $p = P$ gives:

$$P = \left[2h + \frac{2}{3}q^3 - 2\omega\left[q - \frac{q^3}{6}\right]\right]^{1/2}. \tag{13}$$

The expression (7) for energy changes reduces to:

$$\Delta h = \varepsilon \oint \left[1 - \frac{1}{2}q^2\right] \left.\frac{\partial P}{\partial \omega}\right|_{(h_{\mathrm{sep}},\omega_0)} \mathrm{d}q. \tag{14}$$

This can be integrated in closed form to obtain:

$$\Delta h = -\varepsilon\frac{6\sqrt{2a}}{5\sqrt{1+(\omega/2)}}\left[\frac{7}{4}a^4 - 6a^2 + 5\right] \tag{15}$$

where

$$a = \left(\frac{1}{2} + \frac{1}{\omega}\right)^{-1/2} \tag{16}$$

and where Δh is an approximation for the change in energy involved in a single passage around the separatrix loop.

Equation (15) may be used to predict which initial conditions will be captured. Suppose that at $t = 0$, $\omega = 1$. Then from equation (3) the instantaneous saddle will be located at $q = \sqrt{2/3}$, $p = 0$ when $t = 0$. From equation (11), the instantaneous separatrix loop corresponds to energy $h = \sqrt{8/27}$ when $t = 0$. Consider a family of trajectories with the initial conditions $q(0) = \sqrt{2/3}, p(0) < 0$, call it F, see Figure 4. There will be a critical value of $p(0)$, call it p^*, such that motion in the family F with $0 > p(0) > p^*$ will be captured, while motions with $p(0) < p^* < 0$ will escape. The value of p^* will correspond in our energy method to an initial energy which after passage around the loop ends up at the separatrix value of $h = \sqrt{8/27}$. The corresponding initial energy $h^* = \sqrt{8/27} - \Delta h$, where Δh is given by (15) for $\omega = \omega(0) = 1$, i.e., $\Delta h = -\frac{192}{45}[\frac{2}{27}]^{1/4}\varepsilon$. Substituting this value of h^* into equation (11) with $q = q(0) = \sqrt{2/3}$ gives $p^* = -\sqrt{-2\Delta h} = -2.11\sqrt{\varepsilon}$. This theoretical value of p^* is compared to values obtained by numerical integration below:

ε	$p^*_{\text{theoretical}}$	$p^*_{\text{numerical}}$	error
0.001	-0.0667	-0.0666	0.1%
0.01	-0.211	-0.206	2.4%
0.1	-0.667	-0.585	14%

As expected, the agreement is best for small values of ε.

As a final comment on the energy method, we note that expression (15) for Δh is independent of the initial energy h_0. This is due to the replacement of ω and h in the integral (7) or (14) by the values ω_0 and h_{sep}. This approximation is justified by the slow evolution of ω and h, cf. equations (B10) in Appendix B, as compared to the time of passage of a motion around the separatrix, which occurs on a faster time scale. Thus we assume that the motion is close to the separatrix (in order that h_0 be close to h_{sep}). However, in the $\varepsilon = 0$ system the time of passage for motions sufficiently close to the separatrix is arbitrarily large (since the time for passage around the separatrix itself is infinite). In the $\varepsilon > 0$ system this leads to the exclusion of a region of forbidden initial conditions in the neighborhood of the separatrix, of measure $\sqrt{\varepsilon}$ (see [14]).

Invariant Manifold Approach

In this section we offer an alternative approach to the energy method for resonant capture problems which contain a small parameter ε. The idea of the method is to find approximations (i) for the hyperbolic motion (HM) in the $\varepsilon > 0$ system which corresponds to the saddle equilibrium in the $\varepsilon = 0$ frozen system, and then (ii) for the S and U manifolds of the HM. The asymptotic expansions which we present are based on theorems asserting the existence of invariant manifolds for small ε as presented in [18]. These guarantee the existence of a normally hyperbolic motion HM which reduces to the saddle point for $\varepsilon = 0$, and along which motion is slow of $O(\varepsilon)$. The HM is proved to possess S and U manifolds along which points approach HM exponentially fast for small values of ε. Moreover, asymptotic expressions for the S and U manifolds in the form of power series in ε are proved to be uniformly valid on $[t_0, \infty)$ and $(-\infty, t_0]$, respectively.

We take the dynamical system in the form (cf. equations (4)–(6)):

$$\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x}, \omega) + \varepsilon \mathbf{g}(\mathbf{x}, \omega) \tag{17}$$

$$\dot{\omega} = \varepsilon f(\mathbf{x}, \omega) \tag{18}$$

where $\mathbf{x} = (q, p)$, $\mathbf{G}$ and $\mathbf{g}$ are 2-vectors. We assume the $\varepsilon = 0$ system contains a saddle and an associated separatrix on which we have an explicit expression for $\mathbf{x}(t)$. We look for the corresponding HM in the $\varepsilon > 0$ system in the form of an invariant manifold:

$$\mathbf{x} = \varphi(\omega). \tag{19}$$

This procedure generates an expression for the HM because (19) does not contain t, and thus for small ε we are perturbing off of the time independent (equilibrium) solutions of the $\varepsilon = 0$ system. Differentiating (19),

$$\dot{\mathbf{x}} = D\varphi\dot{\omega} \tag{20}$$

where $D\varphi = \partial\varphi/\partial\omega$. Substituting (17)–(19) into (20) gives:

$$\mathbf{G}(\varphi, \omega) + \varepsilon\mathbf{g}(\varphi, \omega) = \varepsilon D\varphi f(\varphi, \omega). \tag{21}$$

Equation (21) may be solved for the invariant manifold $\varphi(\omega)$ by expanding φ in a power series in ε:

$$\varphi(\omega) = \varphi_0(\omega) + \varepsilon\varphi_1(\omega) + \cdots \tag{22}$$

This gives to $O(\varepsilon)$:

$$\mathbf{G}(\varphi_0 + \varepsilon\varphi_1, \omega) + \varepsilon\mathbf{g}(\varphi_0 + \varepsilon\varphi_1, \omega) = \varepsilon(D\varphi_0 + \varepsilon D\varphi_1)f(\varphi_0 + \varepsilon\varphi_1, \omega) \tag{23}$$

$$\mathbf{G}(\varphi_0, \omega) + \varepsilon D\mathbf{G}(\varphi_0, \omega)\varphi_1 + \varepsilon\mathbf{g}(\varphi_0, \omega) = \varepsilon D\varphi_0 f(\varphi_0, \omega) \tag{24}$$

where $D\mathbf{G}(\varphi, \omega) = (\partial\mathbf{G}/\partial\varphi)$. Equating coefficients of ε^0 and ε^1 to zero in (24), we get

$$\mathbf{G}(\varphi_0, \omega) = 0 \tag{25}$$

$$D\mathbf{G}(\varphi_0, \omega)\varphi_1 + \mathbf{g}(\varphi_0, \omega) = D\varphi_0 f(\varphi_0, \omega). \tag{26}$$

Here equation (25) determines $\varphi_0(\omega)$, whereupon equation (26) can be solved for $\varphi_1(\omega)$. Note from equations (17) and (18) that φ_0 given by equation (25) is just the location of the saddle point in the frozen $\varepsilon = 0$ system. Once φ_0 and φ_1 have been found, the time history of the HM may be obtained from equation (18):

$$\dot{\omega} = \varepsilon f(\varphi_0(\omega) + \varepsilon\varphi_1(\omega), \omega). \tag{27}$$

Equation (27) may also be solved using asymptotics,

$$\omega = \omega_0(t) + \varepsilon\omega_1(t) + \cdots \tag{28}$$

which gives

$$\dot{\omega}_0 = 0 \Rightarrow \omega_0 = \text{constant} = \omega(0) \tag{29}$$

$$\dot{\omega}_1 = f(\varphi_0(\omega(0)), \omega(0)) \tag{30}$$

where $\omega(0)$ is the initial value of ω.

Finally, substitution of $\omega(t)$ into (22) and (19) gives $\mathbf{x}(t)$ for the HM. For consistency the resulting expression for $\mathbf{x}(t)$ should be expanded in a power series of ε.

Stable and Unstable Manifolds of the Hyperbolic Motion

In order to obtain S and U manifolds of the HM, we first translate coordinates so that the HM lies at the origin. Let $\mathbf{x}^*(t)$ and $\omega^*(t)$ represent the HM computed in the previous section. Then define $\mathbf{u}$ and v by

$$\mathbf{u} = \mathbf{x} - \mathbf{x}^*(t), \quad v = \omega - \omega^*(t). \tag{31}$$

Substitution of (31) into (17) and (18), and using the fact that $\mathbf{x}^*(t)$ and $\omega^*(t)$ satisfy equations (17) and (18), we obtain

$$\dot{\mathbf{u}} = \mathbf{G}(\mathbf{x}^* + \mathbf{u}, \omega^* + v) - \mathbf{G}(\mathbf{x}^*, \omega^*) + \varepsilon \mathbf{g}(\mathbf{x}^* + \mathbf{u}, \omega^* + v) - \varepsilon \mathbf{g}(\mathbf{x}^*, \omega^*) \tag{32}$$

$$\dot{v} = \varepsilon f(\mathbf{x}^* + \mathbf{u}, \omega^* + v) - \varepsilon f(\mathbf{x}^*, \omega^*). \tag{33}$$

Next we expand $\mathbf{u}$, v, $\mathbf{x}^*$ and ω^* in power series:

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \cdots, \quad v = v_0 + \varepsilon v_1 + \cdots \tag{34}$$

$$\mathbf{x}^* = \mathbf{x}_0^* + \varepsilon \mathbf{x}_1^* + \cdots, \quad \omega = \omega_0^* + \varepsilon \omega_1^* + \cdots \tag{35}$$

Note that $\mathbf{x}_0^*$ and ω_0^* correspond to the HM in the $\varepsilon = 0$ (frozen) case, i.e., to the saddle equilibrium at $t = 0$. This follows because $\omega_0^* = \omega(0)$ from equation (29), and $\mathbf{x}_0^* = \mathbf{x}_0^*(\omega_0^*) = \mathbf{x}_0^*(\omega(0))$ from equation (19). In particular, $\mathbf{x}_0^*$ and ω_0^* are both constants, a fact we shall refer to later on.

Substituting (34) and (35) into (32) and (33), collecting terms and equating coefficients of ε^0 and ε^1 to zero yields equations on $\mathbf{u}_0$, $\mathbf{u}_1$, v_0 and v_1. In particular, the equation on v_0 is simply $\dot{v} = 0$, which means that v_0 is a constant. For motions on the S (or U) manifold, we require that $\mathbf{u}$ and v approach zero as t approaches positive (or negative) infinity. From (34) this means that each of $\mathbf{u}_0$, $\mathbf{u}_1$, v_0 and v_1 must approach zero in these respective limits, and thus

$$v_0 = 0. \tag{36}$$

We find:

$$\dot{\mathbf{u}}_0 = \mathbf{G}(\mathbf{x}_0^* + \mathbf{u}_0, \omega_0^*) - \mathbf{G}(\mathbf{x}_0^*, \omega_0^*) \tag{37}$$

$$\begin{aligned} \dot{\mathbf{u}}_1 = {} & D\mathbf{G}(\mathbf{x}_0^* + \mathbf{u}_0, \omega_0^*)(\mathbf{x}_1^* + \mathbf{u}_1) - D\mathbf{G}(\mathbf{x}_0^*, \omega_0^*)\mathbf{x}_1^* \\ & + \frac{\partial \mathbf{G}}{\partial \omega}(\mathbf{x}_0^* + \mathbf{u}_0, \omega_0^*)(\omega_1^* + v_1) - \frac{\partial \mathbf{G}}{\partial \omega}(\mathbf{x}_0^*, \omega_0^*)\omega_1^* \\ & + \mathbf{g}(\mathbf{x}_0^* + \mathbf{u}_0, \omega_0^*) - \mathbf{g}(\mathbf{x}_0^*, \omega_0^*) \end{aligned} \tag{38}$$

$$\dot{v}_1 = f(\mathbf{x}_0^* + \mathbf{u}_0, \omega_0^*) - f(\mathbf{x}_0^*, \omega_0^*). \tag{39}$$

The desired solution $\mathbf{u}_0$ to equation (37) is just the motion around the separatrix in the $\varepsilon = 0$ frozen system. Since $\mathbf{x}_0^*$ and ω_0^* are both constants, equation (37) is autonomous and $\mathbf{u}_0 = \mathbf{u}_0(t - t_0)$, where t_0 is an arbitrary constant. Assuming $\mathbf{u}_0$ to be known, equation (39) for

v_1 is of the form $\dot{v}_1 = F(t)$, where $F(t)$ is known, and thus v_1 may be obtained by quadrature. Assuming $\mathbf{u}_0$ and v_1 to be known, equation (38) for $\mathbf{u}_1$ is of the form

$$\dot{\mathbf{u}}_1 = A(t)\mathbf{u}_1 + \mathbf{B}(t) \tag{40}$$

where $A(t) = D\mathbf{G}(\mathbf{x}_0^* + \mathbf{u}_0, \omega_0^*)$ is a known 2×2 matrix and $\mathbf{B}(t)$ is a known column vector. The general solution to equation (40) will be of the form:

$$\mathbf{u}_1 = c_1\mathbf{U}_1 + c_2\mathbf{U}_2 + \mathbf{U}_p \tag{41}$$

where $\mathbf{U}_1$ and $\mathbf{U}_2$ are linearly independent complementary solutions and where $\mathbf{U}_p$ is a particular solution. The following observations on the solution of equation (40) are based on an analysis of the forced Duffing equation by Vakakis [20].

$\mathbf{U}_1$ may always be found by differentiating $\mathbf{u}_0$:

$$\mathbf{U}_1 = \dot{\mathbf{u}}_0. \tag{42}$$

This result follows by differentiating equation (37) on $\mathbf{u}_0$, recalling that $\mathbf{x}_0^*$ and ω_0^* are constants:

$$\ddot{\mathbf{u}}_0 = D\mathbf{G}(\mathbf{x}_0^* + \mathbf{u}_0, \omega_0^*)\dot{\mathbf{u}}_0 \tag{43}$$

which is equation (40) with $\mathbf{B}(t) = 0$.

With one solution known, the second complementary solution $\mathbf{U}_2$ and the particular solution $\mathbf{U}_p$ may be found by variation of parameters. The arbitrary constants c_1 and c_2 are expected to be found by requiring $\mathbf{u}_1$ to approach zero as t approaches positive (or negative) infinity on the S (or U) manifold. As pointed out by Vakakis [20], however, c_1 may always be taken as zero, since the c_1 term in (41) corresponds to a time shift. This follows since

$$\mathbf{u}(t) = \mathbf{u}_0(t) + \varepsilon\mathbf{u}_1(t) + \cdots = \mathbf{u}_0(t) + \varepsilon c_1\dot{\mathbf{u}}_0(t) + \cdots \tag{44}$$

$$\mathbf{u}(t) = \mathbf{u}_0(t + \varepsilon c_1). \tag{45}$$

The resulting expressions for $\mathbf{u}$ and v on the S (or U) manifold of the HM represent a two-dimensional surface in the three-dimensional q–p–ω phase space parameterized by t and t_0, where t_0 is the arbitrary constant in $\mathbf{u}_0 = \mathbf{u}_0(t - t_0)$. By choosing t such that $\omega = \omega^* + v$ is a constant K, we may obtain the intersection of the S (or U) manifold with the plane $\omega = K$, i.e. a curve in the q–p plane, parameterized by t_0, which separates those initial conditions which lead to capture from those which escape capture.

Application of Invariant Manifolds to Model Problem

In this section we apply the foregoing approach to equations (8)–(10). Comparison with equations (17) and (18) gives

$$\mathbf{G} = \begin{bmatrix} p \\ q^2 - \omega(1 - \frac{1}{2}q^2) \end{bmatrix}, \quad \mathbf{g} = 0, \quad f = 1 - \frac{1}{2}q^2. \tag{46}$$

We write equations (19) and (22) in the form:

$$\mathbf{x} = \begin{bmatrix} q \\ p \end{bmatrix} = \varphi(\omega) = \begin{bmatrix} Q(\omega) \\ P(\omega) \end{bmatrix} = \begin{bmatrix} Q_0(\omega) + \varepsilon Q_1(\omega) \\ P_0(\omega) + \varepsilon P_1(\omega) \end{bmatrix}. \tag{47}$$

Then equations (25) and (26) become respectively:

$$P_0 = 0, \quad Q_0^2 - \omega \left[1 - \frac{1}{2} Q_0^2\right] = 0 \Rightarrow Q_0 = \left[\frac{1}{\omega} + \frac{1}{2}\right]^{-1/2} \tag{48}$$

$$\begin{bmatrix} 0 & 1 \\ (\omega+2)Q_0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ P_1 \end{bmatrix} = \begin{bmatrix} \frac{\partial Q_0}{\partial \omega} (1 - \frac{1}{2} Q_0^2) \\ 0 \end{bmatrix} \tag{49}$$

which gives

$$Q_1 = 0, \quad P_1 = \frac{\partial Q_0}{\partial \omega} \left(1 - \frac{1}{2} Q_0^2\right). \tag{50}$$

Equation (27) becomes

$$\dot{\omega} = \frac{2\varepsilon}{2+\omega}. \tag{51}$$

Choosing the initial condition on ω as

$$\omega(0) = 1 \tag{52}$$

we find equation (28) to take the form:

$$\omega = 1 + \frac{2}{3} \varepsilon t + \cdots = \omega^*. \tag{53}$$

Substituting (53) into (48) and (50), we obtain the following expressions for the HM:

$$\mathbf{x}^* = \begin{bmatrix} Q^* \\ P^* \end{bmatrix} = \begin{bmatrix} \sqrt{2/3} + \sqrt{8/243}\, \varepsilon t \\ \sqrt{8/243}\, \varepsilon \end{bmatrix} + O(\varepsilon^2). \tag{54}$$

Next we look for the S manifold of this HM. We set

$$\mathbf{u} = \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \mathbf{x} - \mathbf{x}^* = \begin{bmatrix} q - Q^* \\ p - P^* \end{bmatrix}, \quad v = \omega - \omega^*. \tag{55}$$

We expand

$$\xi = \xi_0 + \varepsilon \xi_1 + \cdots, \quad \eta = \eta_0 + \varepsilon \eta_1 + \cdots, \quad v = \varepsilon v_1 + \cdots \tag{56}$$

Substituting (55) and (56) into equations (8)–(10) gives equations (37)–(39), which for convenience we write in the following form:

$$\ddot{\xi}_0 - \frac{3}{2} \xi_0^2 - \sqrt{6}\, \xi_0 = 0 \tag{57}$$

$$\ddot{\xi}_1 - 3(\xi_0 + \sqrt{6})\xi_1 = 3\xi_0 Q_1^* + \left(\frac{1}{2} \xi_0^2 + \frac{1}{3} \sqrt{6}\, \xi_0\right)(\omega_1^* + v_1) - \frac{2}{3} v_1 \tag{58}$$

$$\dot{v}_1 = -\frac{1}{2} \xi_0^2 - \sqrt{\frac{2}{3}}\, \xi_0 \tag{59}$$

where from (53) and (54) we get

$$Q_1^* = \sqrt{8/243}\, t, \quad \omega_1^* = \frac{2}{3}\, t. \tag{60}$$

Since we seek the S manifold of the HM, the appropriate solution to (57) corresponds to motion on the separatrix:

$$\xi_0 = \sqrt{6}\,(\tanh^2 \beta\tau - 1), \quad \beta = (3/8)^{1/4}, \quad \tau = t - t_0. \tag{61}$$

Substituting (61) into (59) and using the condition that v_1 should approach zero as t approaches infinity gives:

$$v_1 = -\frac{\sinh \beta\tau}{\beta \cosh^3 \beta\tau}\,. \tag{62}$$

Equation (58) becomes:

$$\ddot{\xi}_1 - \sqrt{6}\left[1 - \frac{3}{\cosh^2 \beta\tau}\right] \xi_1 = N(\tau) \tag{63}$$

where

$$N(\tau) = \frac{\tau + t_0}{\cosh^2 \beta\tau}\left[\frac{2}{\cosh^2 \beta\tau} - \frac{8}{3}\right] + \frac{\sinh \beta\tau}{\beta \cosh^3 \beta\tau}\left[\frac{2}{3} + \frac{2}{\cosh^2 \beta\tau} - \frac{3}{\cosh^4 \beta\tau}\right]. \tag{64}$$

As noted previously, the general solution to (63) takes the form:

$$\xi_1 = c_1 U_1 + c_2 U_2 + U_p \tag{65}$$

in which, from (42) and (61), we may take

$$U_1 = \frac{\sinh \beta\tau}{\cosh^3 \beta\tau}\,. \tag{66}$$

In order to find U_2 and U_p we used variation of parameters and the computer algebra system MACSYMA. Expressions for U_2 and U_p are given in Appendix C.

As discussed in the previous section, we may take $c_1 = 0$ since it represents an arbitrary phase shift, see equations (44) and (45). In order to obtain the value of c_2, we must obtain the large t asymptotic behavior of U_2 and U_p, choosing c_2 so that ξ_1 approaches zero as t approaches infinity. Based on the expressions given in Appendix C, we find that in the limit as $t \to \infty$,

$$U_2 \simeq \frac{1}{16\beta}\, e^{2\beta\tau}, \quad U_p \simeq -\frac{4\tau^2}{3\beta}\, e^{-2\beta\tau} \tag{67}$$

and therefore we choose

$$c_2 = 0. \tag{68}$$

From equations (53)–(56), (61), (62), (65) and (68), we obtain the following expressions for $q(t)$ and $\omega(t)$ on the S manifold of the HM:

$$q = \sqrt{2/3} + \sqrt{8/243}\, \varepsilon t + \sqrt{6}(\tanh^2 \beta(t - t_0) - 1) + \varepsilon U_p(t) + O(\varepsilon^2) \tag{69}$$

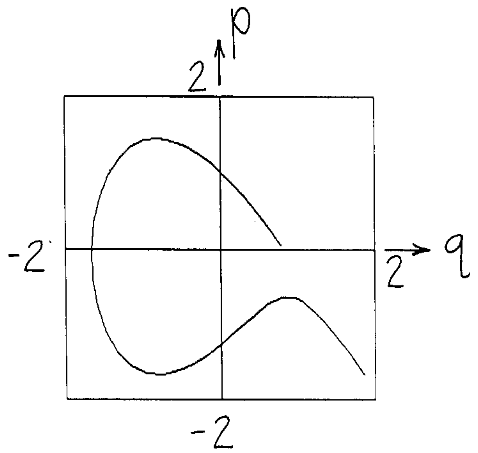

Fig. 5. Analytic approximation of the S manifold of the HM as obtained by invariant manifold theory and perturbations. The intersection of the S manifold with the plane $\omega = 1$ is shown for $\varepsilon = 0.1$. Compare with the results of numerical integration in Figure 4.

$$\omega = 1 + \frac{2}{3}\varepsilon t + \varepsilon \frac{\sinh \beta(t - t_0)}{\beta \cosh^3 \beta(t - t_0)} + O(\varepsilon^2) \tag{70}$$

where $U_p(t)$ is given in Appendix C. From equation (8), an expression for $p = \dot{q}$ may be obtained by differentiating (69).

Here the S manifold of the HM is seen as a two-dimensional surface in the three-dimensional q–p–ω phase space parameterized by t and t_0. It is interesting to compare these results with the numerical results of Figure 4, which effectively shows the intersection of the S manifold with the plane $\omega = 1$ (since Figure 4 was generated using the initial condition $\omega(0) = 1$). In order to obtain the comparable curve analytically, we set $\omega = 1$ in equation (70) and denote by t_1 the corresponding time, which may be related to the parameter t_0 as follows:

$$t_0 = -(t_1 - t_0) - \frac{3}{2}\frac{\sinh \beta(t_1 - t_0)}{\beta \cosh^3 \beta(t_1 - t_0)} \equiv F(t_1 - t_0). \tag{71}$$

Now if we set $t = t_1$ in $q(t; t_0)$ we obtain

$$q(t_1; t_0) = q(t_0 + (t_1 - t_0); t_0) = q(F(t_1 - t_0) + (t_1 - t_0); F(t_1 - t_0)). \tag{72}$$

That is, q becomes a function of only the single argument $t_1 - t_0$. By varying this parameter we may obtain point which lie on the intersection of the S manifold and the plane $\omega = 1$, see Figure 5. We note that there is reasonable agreement between our approximate analytic expressions and the numerical results of Figure 4.

Conclusions

We have, in the context of a physical example (Figure 1), derived a simplified model of resonant capture (equations (1) and (2)). The simplified model lies in the neighborhood of a resonance manifold in the phase space of the physical example. (The resonance manifold may be described as that region where averaging fails, Appendix A.) For small values of the parameter ε, the simplified model may be viewed as a nonlinear dynamical system with a separatrix, which is driven by a slowly changing feedback variable. Capture corresponds to those trajectories which 'cross' the separatrix.

We have treated the model problem in two ways. Firstly we used an energy approach which derives (Appendix B) an approximation for the change in energy in the $\varepsilon > 0$ (perturbed) system for motions which lie close to the separatrix loop in the $\varepsilon = 0$ (frozen) system. The energy approach has been previously used [11, 12] to calculate the probability of capture of a random initial condition and lends itself easily to numerical computation. However, this approach offers very limited information regarding which initial conditions will lead to capture. The main drawbacks of the energy approach are (a) its lack of information about the dependence of the dynamics on phase (which may be remedied by extending the method [7]), and (b) the difficulty in extending the approach beyond the lowest order in ε [8].

In contrast to the energy approach, we presented a new approach based on invariant manifold theory [18]. The saddle equilibrium in the frozen system is replaced by a hyperbolic motion (HM) in the perturbed system. A perturbation method is used to obtain an approximation for the HM. Although we worked to $O(\varepsilon)$ for simplicity of presentation, the perturbation series is easily extended to terms of $O(\varepsilon^n)$.

Once the HM has been found, another perturbation method may be used to find its stable (S) and unstable (U) manifolds. These are surfaces in the perturbed system which correspond to the separatrix loop in the frozen system. In particular, the S manifold separates those motions which get captured from those that pass through the resonance. Our perturbation procedure utilizes an approach used by Vakakis [20] in a study of Duffing's equation.

The invariant manifold approach does not suffer from the drawbacks mentioned above in connection with the energy approach. Moreover, the mathematics involved in the construction of the asymptotic expressions is elementary, in contrast to other recent treatments of separatrix crossing which have involved perturbation schemes based on elliptic functions [9, 10, 17]. However, the calculations, while elementary, are very long (Appendix C), and are best performed using computer algebra.

It should be noted that the model problem is exceptionally uncomplicated because the separatrix grows monotonically, leading to the two possibilities of capture or pass-through. In problems where this is not the case, e.g. if the size of the separatrix varies periodically [6], repeated separatrix crossings can lead to multiple escapes and reentries. In this case we would expect the Melnikov integral to exhibit zeroes, and the study of the accompanying chaos can be facilitated by the use of *turnstiles* [1].

Appendix A: Derivation of Equations (1) and (2)

Consider a mechanical system consisting of an unbalanced rotor attached to an elastic support and driven by a constant torque as in Figure 1. Neglecting gravity and friction, the following dimensionless form of the equations of motion have been derived in [17]:

$$z_{TT} + z = e\theta_T^2 \sin\theta - e\theta_{TT}\cos\theta = e\theta_T^2 \sin\theta + O(e^2) \tag{A1}$$

$$\theta_{TT} = eK - ez_{TT}\cos\theta = eK + ez\cos\theta + O(e^2) \tag{A2}$$

where $e \ll 1$ is the eccentricity of the unbalanced rotor mass, and where the applied torque eK has been assumed to be of order e. As in [17], we transform to polar coordinates

$$z = r\sin\psi, \quad z_T = r\cos\psi \tag{A3}$$

giving the first order system

$$r_T = e\Omega^2 \sin\theta \cos\psi \tag{A4}$$

$$\psi_T = 1 - \frac{e}{r}\,\Omega^2 \sin\theta \sin\psi \tag{A5}$$

$$\theta_T = \Omega \tag{A6}$$

$$\Omega_T = eK + er\cos\theta\sin\psi. \tag{A7}$$

First order averaging of (A4)–(A7) gives:

$$r_T = -\frac{e}{2}\,\Omega^2 \sin(\psi - \theta) \tag{A8}$$

$$\Omega_T = eK + \frac{er}{2}\,\sin(\psi - \theta) \tag{A9}$$

$$\theta_T = \Omega \tag{A10}$$

$$\psi_T = 1 - \frac{e}{2r}\,\Omega^2 \cos(\psi - \theta). \tag{A11}$$

Note in equations (A8)–(A11) that we have only performed 'partial averaging', i.e., we have not removed the trig terms with argument $\psi - \theta$. This is because the near-identity transformation upon which averaging is based has vanishing denominators at $\Omega = 1$, the resonance manifold [17].

Equations (A8)–(A11) may be reduced to a system of three equations by setting:

$$q = \theta - \psi - \frac{\pi}{2} \tag{A12}$$

which gives

$$r_T = \frac{e}{2}\,\Omega^2 \cos q \tag{A13}$$

$$\Omega_T = eK - \frac{er}{2}\,\cos q \tag{A14}$$

$$q_T = \Omega - 1 + \frac{e}{2r}\,\Omega^2 \sin q. \tag{A15}$$

In order to investigate dynamics in the neighborhood of the resonance manifold $\Omega = 1$, we set $\Omega = 1 + O(e)$. Then differentiating the last equation gives to $O(e^2)$:

$$q_{TT} + \frac{er}{2}\,\cos q = eK \tag{A16}$$

$$r_T = \frac{e}{2}\,\cos q. \tag{A17}$$

For small e, r is nearly a constant. For r constant, the nature of the phase portrait of the q equation in the q–q_T phase plane depends on whether r is larger or smaller than $2K$. As r increases through $2K$, a pair of equilibrium points bifurcate out of $q = 0$. One of these equilibria is a center, while the other is a saddle. The saddle is associated with a separatrix which surrounds the center. Motions which start inside the separatrix correspond to 'capture' while motions which stay outside the separatrix correspond to 'pass-through'. In order to investigate the capture process for small e, we set in (A16) and (A17)

$$r = 2K\left(1 + \frac{\omega}{2}\right), \quad T = \sqrt{\frac{2}{eK}}\, t, \quad \varepsilon = \sqrt{\frac{e}{2K^3}} \ll 1.$$

For convenience in the ensuing calculations, we replace $\cos q$ in (A16) and (A17) by the first two terms of its Taylor series, $1 - (q^2/2)$. This replaces the actual system with the model problem of equations (1) and (2) of the text which is qualitatively similar to it. Both systems contain a separatrix loop which grows in size, capturing motions which pass close to it. The main effect of the truncation of $\cos q$ is to eliminate the periodicity of the phase space, a feature which is not important here.

Appendix B: Derivation of Equation (7)

We will begin our derivation with systems of the following form:

$$\frac{dq}{dt} = \xi_1(q, p, \omega) \tag{B1a}$$

$$\frac{dp}{dt} = \xi_2(q, p, \omega) \tag{B1b}$$

$$\frac{d\omega}{dt} = \varepsilon f(q, p, \omega). \tag{B1c}$$

In order to obtain an expression for the energy change along an orbit we will begin by changing the independent variable from t to q, such that $p = p(q)$, $\omega = \omega(q)$, and $t = t(q)$. Using the chain rule:

$$\frac{dp}{dq} = \frac{dp}{dt}\frac{dt}{dq}, \quad \frac{dq}{dq} = 1 = \frac{dq}{dt}\frac{dt}{dq}, \quad \frac{d\omega}{dq} = \frac{d\omega}{dt}\frac{dt}{dq}$$

which gives:

$$\frac{dp}{dq} = \frac{dp/dt}{dq/dt} = \frac{\xi_2}{\xi_1}, \quad \frac{dt}{dq} = \frac{1}{dq/dt} = \frac{1}{\xi_1}, \quad \frac{d\omega}{dq} = \frac{d\omega/dt}{dq/dt} = \frac{\varepsilon f}{\xi_1}.$$

We now assume the system to be in near-Hamiltonian form with Hamiltonian $H(q, p, \omega)$ such that:

$$\begin{aligned} \xi_1(q, p, \omega) &= \frac{\partial H}{\partial p}(q, p, \omega) + \varepsilon g_1(q, p, \omega) \\ \xi_2(q, p, \omega) &= -\frac{\partial H}{\partial q}(q, p, \omega) + \varepsilon g_2(q, p, \omega). \end{aligned}$$

Note that the Hamiltonian of the system is assumed to be time-independent. Written in terms of the new independent variable q, our system becomes:

$$\frac{dp}{dq} = \frac{-(\partial H/\partial q) + \varepsilon g_2}{(\partial H/\partial p) + \varepsilon g_1} \tag{B2a}$$

$$\frac{\mathrm{d}t}{\mathrm{d}q} = \frac{1}{(\partial H/\partial p) + \varepsilon g_1} \tag{B2b}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}q} = \frac{\varepsilon f}{(\partial H/\partial p) + \varepsilon g_1}\,. \tag{B2c}$$

To facilitate the calculation of the change of energy, we follow Cary *et al.* [8] and introduce a new dependent variable $h = h(q)$ to replace $p = p(q)$:

$$h(q) = H(q, p(q), \omega(q)). \tag{B3}$$

This new variable is nominally the energy of the system at an instantaneous state. Differentiating equation (B3) with respect to q gives:

$$\frac{\mathrm{d}h}{\mathrm{d}q} = \frac{\partial H}{\partial q} + \frac{\partial H}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}q} + \frac{\partial H}{\partial \omega}\frac{\mathrm{d}\omega}{\mathrm{d}q} \tag{B4}$$

where $\mathrm{d}p/\mathrm{d}q$ and $\mathrm{d}\omega/\mathrm{d}q$ are given by equations (B2). We have transformed the original system (B1) on $q(t), p(t), \omega(t)$ to an equivalent system in terms of $h(q), t(q), \omega(q)$ as given below:

$$\frac{\mathrm{d}h}{\mathrm{d}q} = \frac{\partial H}{\partial q} + \frac{\partial H}{\partial p}\frac{-(\partial H/\partial q) + \varepsilon g_2}{(\partial H/\partial p) + \varepsilon g_1} + \frac{\partial H}{\partial \omega}\frac{\varepsilon f}{(\partial H/\partial p) + \varepsilon g_1} \tag{B5a}$$

$$\frac{\mathrm{d}t}{\mathrm{d}q} = \frac{1}{(\partial H/\partial p) + \varepsilon g_1} \tag{B5b}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}q} = \frac{\varepsilon f}{(\partial H/\partial p) + \varepsilon g_1}\,. \tag{B5c}$$

Thus the change in energy can be found exactly by integrating equation (B5a) along the motion of interest. However, this system is no simpler than the original system. The variables $\{\omega, t, h\}$ are all dependent on q, and p is still present although eliminated as a dependent variable. To simplify, we expand equations (B5) in a Taylor series in ε about $\varepsilon = 0$:

$$\frac{\mathrm{d}h}{\mathrm{d}q} = \varepsilon\left[g_1\frac{\partial H/\partial q}{\partial H/\partial p} + g_2 + f\frac{\partial H/\partial \omega}{\partial H/\partial p}\right] + \cdots \tag{B6a}$$

$$\frac{\mathrm{d}t}{\mathrm{d}q} = \frac{1}{\partial H/\partial p}\left[1 - \varepsilon\frac{g_1}{\partial H/\partial p} + \cdots\right] \tag{B6b}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}q} = \frac{\varepsilon f}{\partial H/\partial p}\left[1 - \varepsilon\frac{g_1}{\partial H/\partial p} + \cdots\right]. \tag{B6c}$$

The variable $p(q)$ can be removed from the system by solving $H = h$ for p:

$$p = P = P(h, \omega, q). \tag{B7a}$$

Substituting (B7a) back into $H = h$ gives an identity

$$h = H(P(h, \omega, q), \omega, q). \tag{B7b}$$

Differentiating (B7a) and (B7b) with respect to q gives

$$\frac{\mathrm{d}P}{\mathrm{d}q} = \frac{\partial P}{\partial h}\frac{\mathrm{d}h}{\mathrm{d}q} + \frac{\partial P}{\partial q} + \frac{\partial P}{\partial \omega}\frac{\mathrm{d}\omega}{\mathrm{d}q} \tag{B8a}$$

$$\frac{\mathrm{d}h}{\mathrm{d}q} = \frac{\partial H}{\partial q} + \frac{\partial H}{\partial p}\frac{\mathrm{d}P}{\mathrm{d}q} + \frac{\partial H}{\partial \omega}\frac{\mathrm{d}\omega}{\mathrm{d}q}\,. \tag{B8b}$$

Substituting (B8a) into (B8b) and collecting terms results in:

$$0 = \left[\frac{\partial H}{\partial \omega} + \frac{\partial H}{\partial p}\frac{\partial P}{\partial \omega}\right]\frac{\mathrm{d}\omega}{\mathrm{d}q} + \left[-1 + \frac{\partial H}{\partial p}\frac{\partial P}{\partial h}\right]\frac{\mathrm{d}h}{\mathrm{d}q} + \left[\frac{\partial H}{\partial q} + \frac{\partial H}{\partial p}\frac{\partial P}{\partial q}\right].$$

Since this is an identity, the bracketed terms must vanish, giving:

$$\frac{\partial P}{\partial h} = \frac{1}{\partial H/\partial p}, \quad \frac{\partial P}{\partial q} = -\frac{\partial H/\partial q}{\partial H/\partial p}, \quad \frac{\partial P}{\partial \omega} = -\frac{\partial H/\partial \omega}{\partial H/\partial p}\,. \tag{B9}$$

Substituting equations (B9) into equations (B6), we obtain

$$\frac{\mathrm{d}h}{\mathrm{d}q} = \varepsilon\left[g_1\frac{\partial P}{\partial q} + g_2 - f\frac{\partial P}{\partial \omega}\right] + \cdots \tag{B10a}$$

$$\frac{\mathrm{d}t}{\mathrm{d}q} = \frac{\partial P}{\partial h} - \varepsilon g_1\left[\frac{\partial P}{\partial h}\right]^2 + \cdots \tag{B10b}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}q} = \varepsilon f\left[\frac{\partial P}{\partial h} - \varepsilon g_1\left[\frac{\partial P}{\partial h}\right]^2 + \cdots\right]. \tag{B10c}$$

Thus the approximate change in energy Δh along any orbit can be found by integrating equation (B10a). Since ω is slowly varying ($\mathrm{d}\omega/\mathrm{d}q$ is $O(\varepsilon)$), we approximate ω by its initial value over the range of integration. Similarly, we hold h fixed during the integration since $\mathrm{d}h/\mathrm{d}q = O(\varepsilon)$. This results in equation (7) of the text.

We note that the foregoing strategy for deriving equation (7) was mentioned, but not published in [8]. The thesis of Bridge [5] contains a similar derivation for the case in which $f(q, p, \omega, t) = 1$ in equation (B1c). A paper by Neishtadt [16] contains an equivalent result derived in a different fashion, namely by using the Melnikov integral.

Appendix C: Expressions for U_2 and U_p in Equation (65) for the Stable Manifold of the HM

Since a complementary solution U_1 to equation (63) is obtainable by differentiating (61),

$$U_1 = \sinh(z)/\cosh(z)\hat{}3 \tag{C1}$$

where $z = (3/8)\hat{}(1/4) * (t - t0)$, we may look for a second linearly independent complementary solution U_2 in the form $U_2(t) = \psi(t)U_1(t)$. Using MACSYMA we obtain:

$$\begin{aligned} U_2 = {} & e\hat{} - (2^*z) * \\ & (e\hat{}(10^*z) + 15^*e\hat{}(8^*z) + (120^*z - 16)^*e\hat{}(6^*z) \\ & - (120^*z) + 144)^*e\hat{}(4^*z) + 15^*e\hat{}(2^*z) + 1) \\ & /(16^*(e\hat{}(2^*z) + 1)\hat{}3). \end{aligned} \tag{C2}$$

In order to obtain a particular solution U_p to (63) we write

$$U_p(t) = \alpha(t)U_1(t) + \beta(t)U_2(t) \tag{C3}$$

and obtain using MACSYMA:

$$\begin{aligned} U_p = {} & 4^*2\hat{}(1/4)^*3\hat{}(1/4)^*e\hat{} - (2^*z)^* \\ & (120^*z^*e\hat{}(10^*z)^*z0 + 30^*e\hat{}(10^*z)^*z0 \\ & + 120^*z^*e\hat{}(8^*z)^*z0 - 90^*e\hat{}(8^*z)^*z0 \\ & - 120^*z^*e\hat{}(6^*z)^*z0 - 270^*e\hat{}(6^*z)^*z0 - 120^*z^*e\hat{}(4^*z)^*z0 \\ & - 150^*e\hat{}(4^*z)^*z0 + 60^*z\hat{}2^*e\hat{}(10^*z) + 60^*z\hat{}2^*e\hat{}(8^*z) \\ & - 120^*z^*e\hat{}(8^*z) + 40^*e\hat{}(8^*z) - 60^*z\hat{}2^*e\hat{}(6^*z) \\ & - 240^*z^*e\hat{}(6^*z) - 265^*e\hat{}(6^*z) - 60^*z\hat{}2^*e\hat{}(4^*z) \\ & - 120^*z^*e\hat{}(4^*z) - 49^*e\hat{}(4^*z) + 17^*e\hat{}(2^*z) + 1) \\ & /(135^*(e\hat{}(2^*z) + 1)\hat{}5) \end{aligned} \tag{C4}$$

where $z0 = (3/8)\hat{}(1/4)^*t0$.

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