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A QUASIPERIODIC MATHIEU EQUATION

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ABSTRACT

In this work we investigate the following quasiperiodic Mathieu equation:

$$\ddot{x} + (\delta + \epsilon \cos t + \epsilon \cos \omega t)\, x = 0$$

We use numerical integration to determine regions of stability in the $\delta - \omega$ plane for fixed ϵ. Graphs of these stability regions are presented, based on extensive computation. In addition, we use perturbations to obtain approximations for the stability regions near $\delta = \frac{1}{4}$ for small ω, and we compare the results with those of direct numerical integration.

INTRODUCTION

A standard procedure for investigating the stability of a periodic motion in an autonomous nonlinear system involves linearizing in the neighborhood of the periodic motion, resulting in a linear differential equation with periodic coefficients, i.e. Floquet theory. The paradigm example is given by Mathieu's equation:

$$\ddot{x} + (\delta + \epsilon \cos t)\, x = 0 \tag{1}$$

In order to investigate the stability of a quasiperiodic (QP) motion in an autonomous nonlinear system, the foregoing approach leads to a linear differential equation with QP coefficients. This subject has received little attention in the literature compared with that given to Floquet theory. E.g., Arnold (1983, p.198) writes, "This theory [of normal forms for nonlinear systems with quasiperiodic coefficients] is unsatisfactory because of the imperfection of the theory of linear equations with quasiperiodic coefficients."

In this work we investigate an example of such a system, the following QP Mathieu equation:

$$\ddot{x} + (\delta + \epsilon \cos t + \epsilon \cos \omega t)\, x = 0 \tag{2}$$

We use numerical integration to determine regions of stability in the $\delta - \omega$ plane for fixed ϵ. Graphs of these stability regions are presented, based on extensive computation. In addition, we use perturbations to obtain approximations for the stability regions near $\delta = \frac{1}{4}$ for small ω, and we compare the results with those of direct numerical integration.

This equation has been referred to in (Davis and Rosenblatt, 1980) and (Vrscay, 1991). Neither of these references offers any numerical results, however.

NUMERICAL INTEGRATION

The linear differential equation (2) will be said to be stable if all solutions remain bounded as $t \to \infty$, and unstable if an unbounded solution exists. We determine stability approximately by the following numerical criterion:

For a given initial condition $(x(0), \dot{x}(0))$, we numerically integrate forward in time for 1000 time units. At each time step, we compute $r(t) = \sqrt{x(t)^2 + \dot{x}(t)^2}$. A point $(\epsilon, \delta, \omega)$ is declared unstable if $r(t) > 10^6\, r(0)$ for

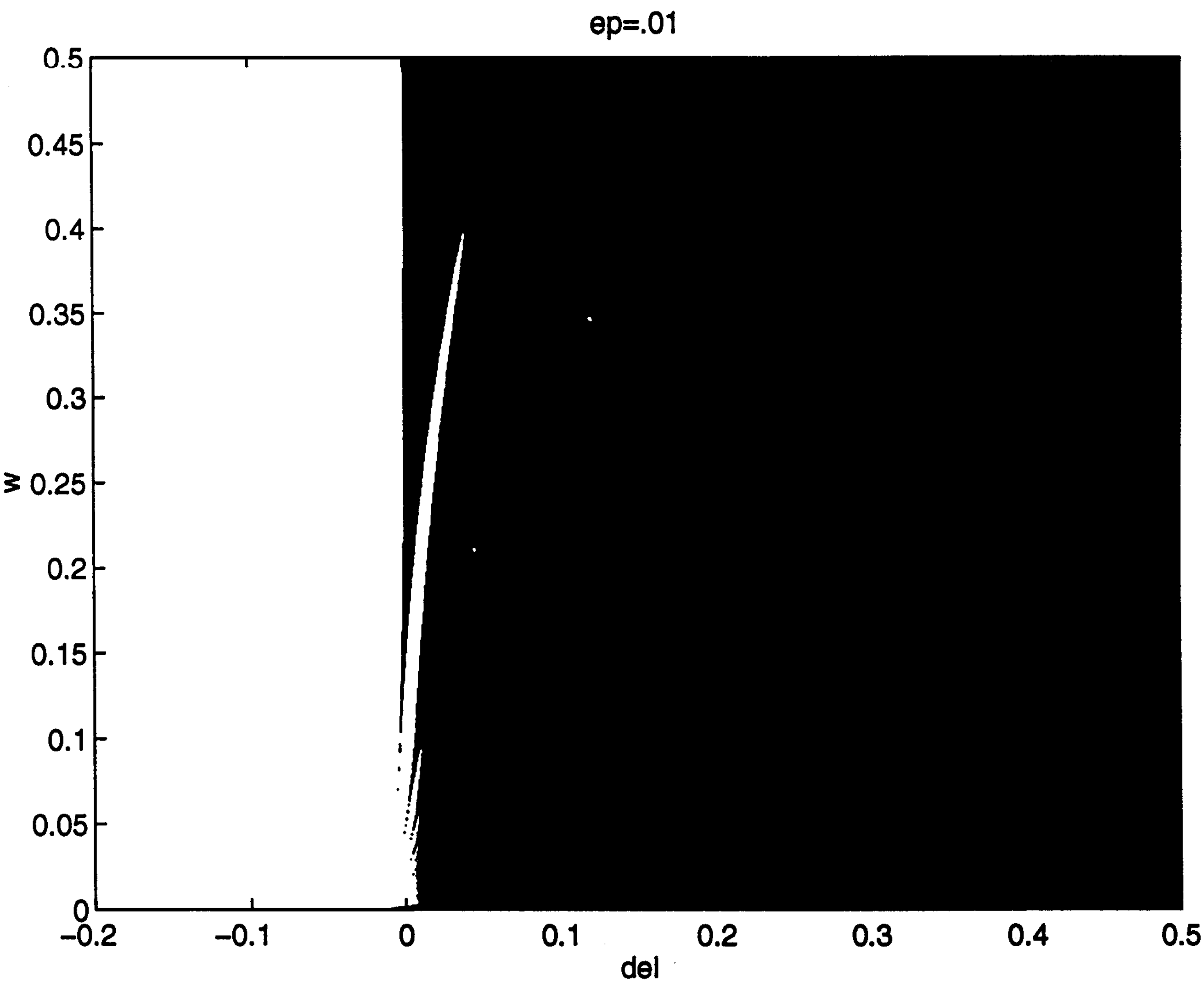

Fig.1. Stability of Eq. (2) for $\epsilon = 0.01$. Black=stable, white=unstable.

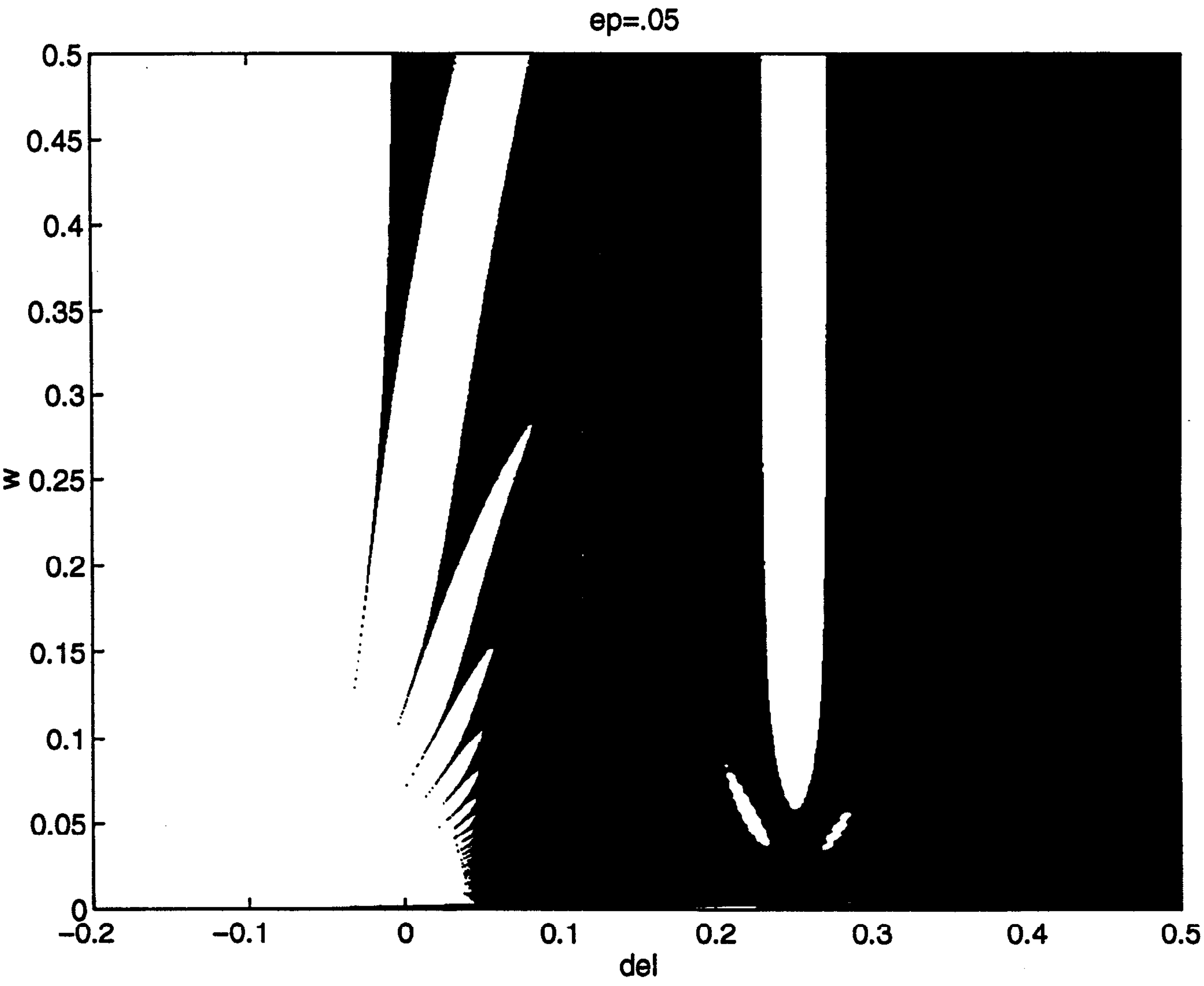

Fig.2. Stability of Eq. (2) for $\epsilon = 0.05$. Black=stable, white=unstable.

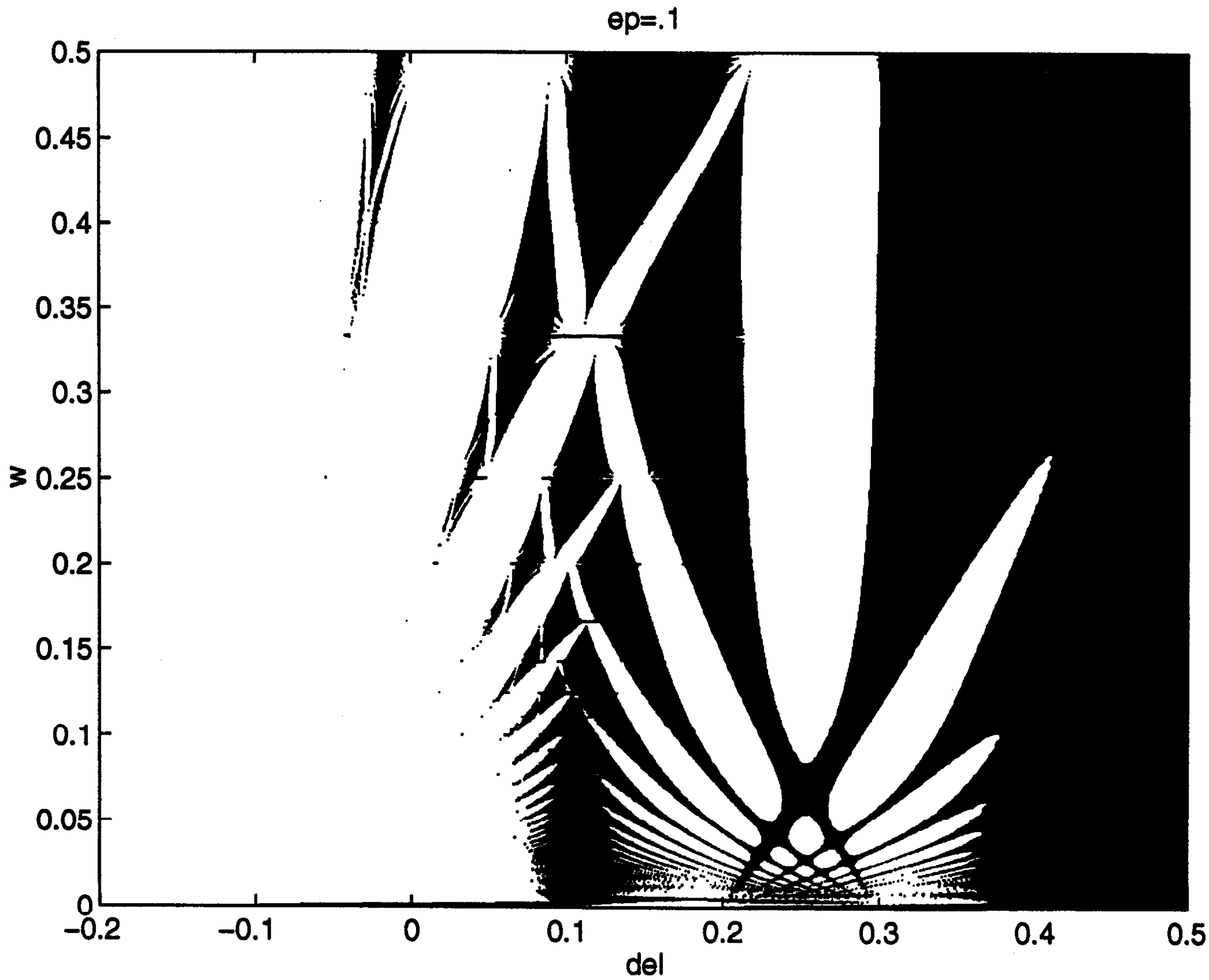

Fig.3. Stability of Eq. (2) for $\epsilon = 0.1$. Black=stable, white=unstable.

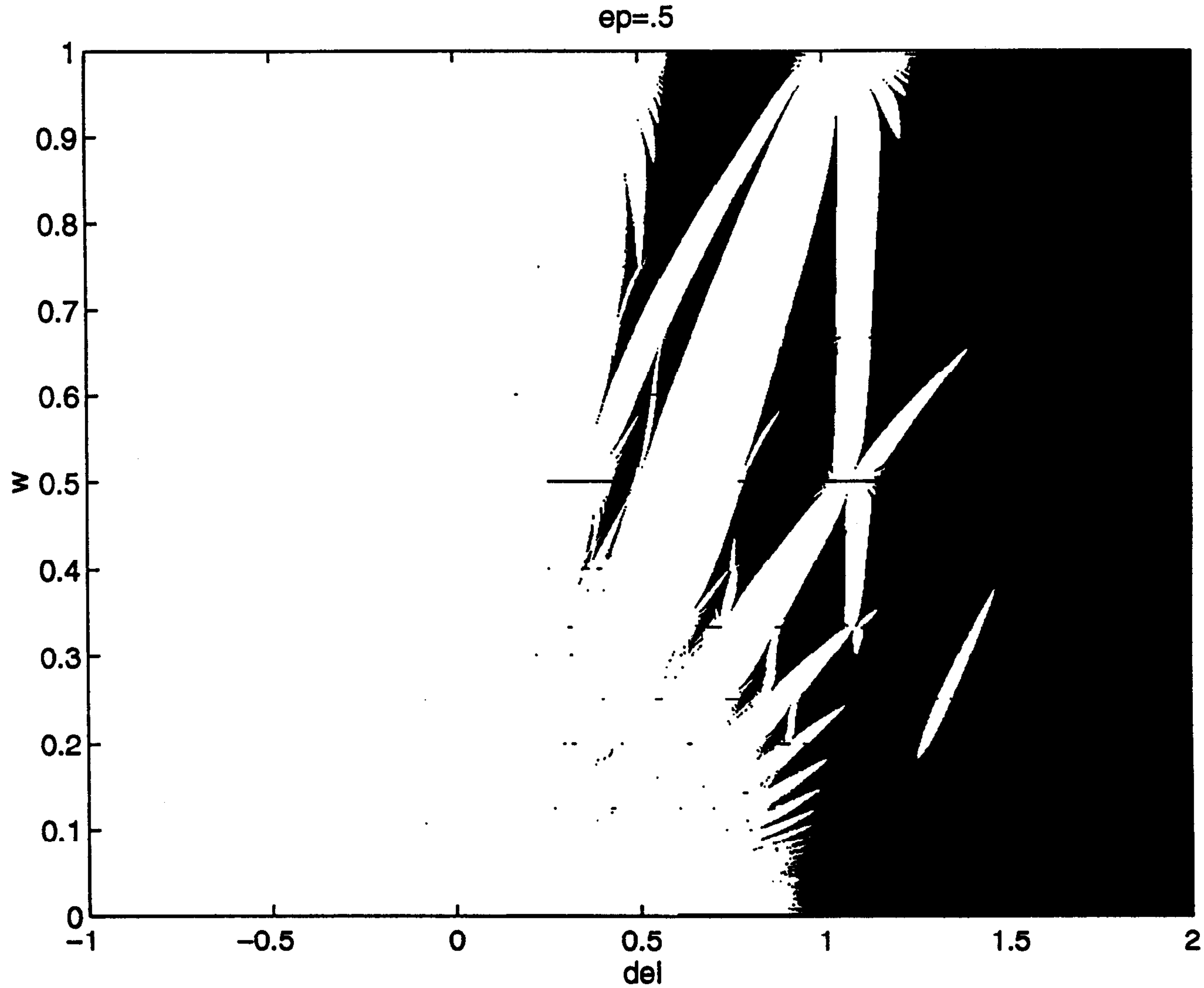

Fig.4. Stability of Eq. (2) for $\epsilon = 0.5$. Black=stable, white=unstable.

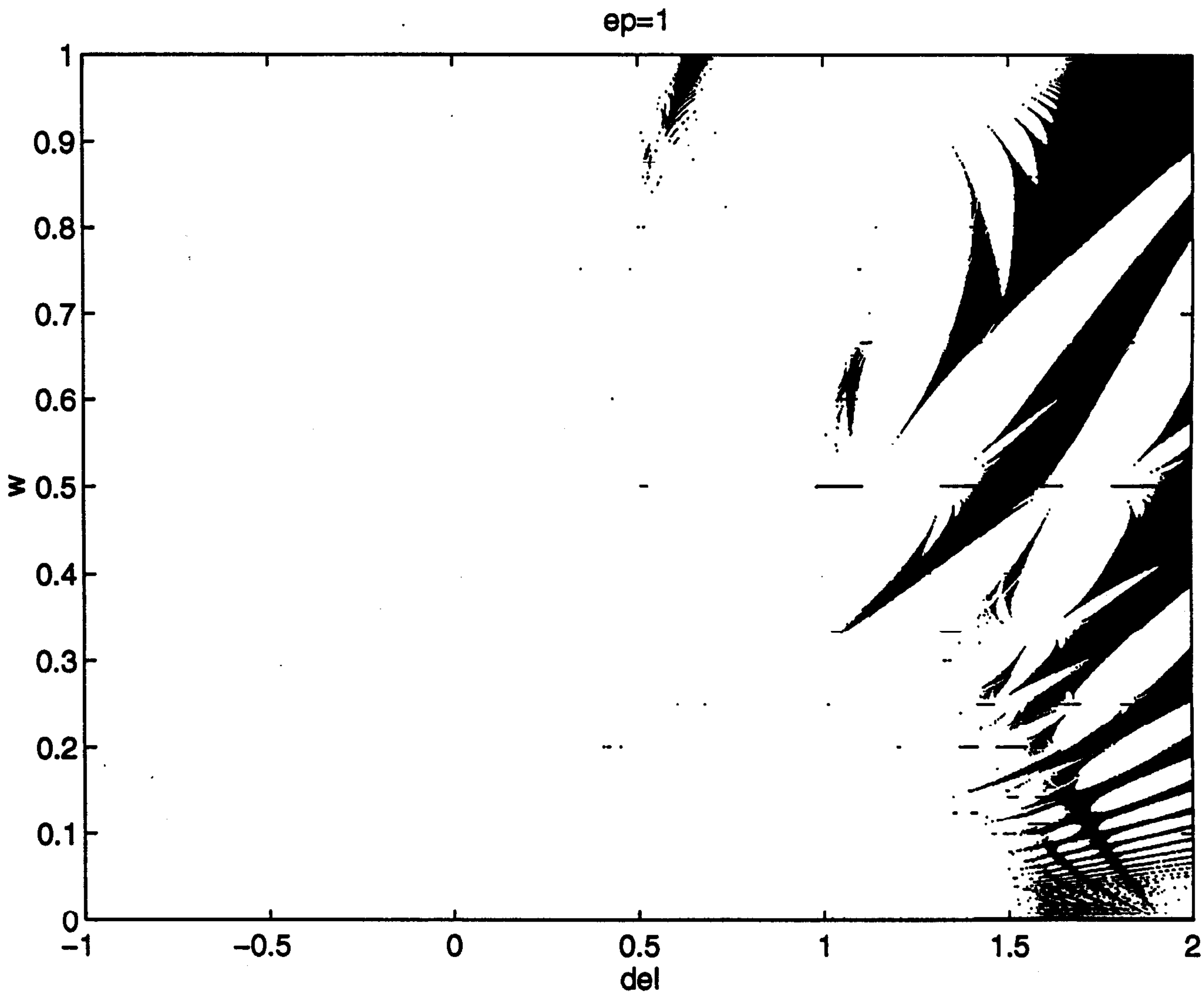

Fig.5. Stability of Eq. (2) for $\epsilon = 1.0$. Black=stable, white=unstable.

any t between 0 and 1000, and stable otherwise. That is, we require a million-fold increase in amplitude in order for a motion to be declared unstable. Our choice of 1000 time units for the duration of the test is a practical compromise between the desirable duration of infinity and reality.

The results of our numerical stability analysis are presented in Figs.1-5 in which regions of stability in the $\delta - \omega$ plane are displayed for $\epsilon = 0.01, 0.05, 0.1, 0.5$ and 1.0, respectively. Note that the scale in Figs.4 and 5 is different from that in Figs.1-3. In all these Figures unshaded regions represent unstable behavior, and shaded regions represent stable behavior. We note that the structure of the stability regions for the QP Mathieu equation(2) is much more complicated than for the Mathieu equation(1) (Stoker, 1950).

PERTURBATION METHOD

In order to better understand the nature of the dynamics of Eq. (2), we present the following perturbation method, valid for small values of the parameter ω. We set

$$\omega = k\epsilon \quad (3)$$

and we use the two-variable expansion method (Bender and Orszag, 1978) in which $\xi = t$ and $\eta = \epsilon t$, whereupon Eq. (2) becomes:

$$x_{\xi\xi} + 2\epsilon x_{\xi\eta} + \epsilon^2 x_{\eta\eta} + (\delta + \epsilon \cos \xi + \epsilon \cos k\eta)\, x = 0 \quad (4)$$

Expanding $x = x_0 + \epsilon\, x_1 + \cdots$ and $\delta = \delta_0 + \epsilon\, \delta_1 + \cdots$ and collecting terms, we obtain

$$x_{0\xi\xi} + \delta_0\, x_0 = 0 \quad (5)$$

$$x_{1\xi\xi} + \delta_0\, x_1 = -2\, x_{0\xi\eta} - \delta_1\, x_0 - (\cos \xi + \cos k\eta)\, x_0 \quad (6)$$

Eq. (5) has the solution:

$$x_0 = R \cos(\sqrt{\delta_0}\xi + \theta) \quad (7)$$

where R and θ are functions of slow time η. Substituting Eq. (7) into Eq. (6), we obtain:

$$x_{1\xi\xi} + \delta_0\, x_1 =$$

$$2\sqrt{\delta_0}\, [\frac{dR}{d\eta} \sin(\sqrt{\delta_0}\xi + \theta) + R\, \frac{d\theta}{d\eta} \cos(\sqrt{\delta_0}\xi + \theta)]$$

$$-\delta_1\, R \cos(\sqrt{\delta_0}\xi + \theta) - R \cos k\eta \cos(\sqrt{\delta_0}\xi + \theta) \quad (8)$$

$$-\frac{R}{2}\, [\cos((\sqrt{\delta_0} + 1)\xi + \theta) + \cos((\sqrt{\delta_0} - 1)\xi + \theta)]$$

Removal of secular terms dictates that we set to zero the coefficients of $\sin(\sqrt{\delta_0}\xi + \theta)$ and $\cos(\sqrt{\delta_0}\xi + \theta)$, giving:

$$\frac{dR}{d\eta} = 0, \qquad 2\sqrt{\delta_0}\, R\frac{d\theta}{d\eta} - \delta_1\, R - R \cos k\eta = 0 \quad (9)$$

The equation $\frac{dR}{d\eta} = 0$ means R is constant and no instability can occur. However, if δ_0 is assigned the resonant value of $\frac{1}{4}$, the term $\cos((\sqrt{\delta_0} - 1)\xi + \theta)$ in Eq. (8) becomes secular:

$$\cos((\sqrt{\delta_0} - 1)\xi + \theta) = \cos(\frac{\xi}{2} - \theta)$$

$$= \cos(\frac{\xi}{2} + \theta - 2\theta)$$

$$= \cos(\frac{\xi}{2} + \theta) \cos 2\theta + \sin(\frac{\xi}{2} + \theta) \sin 2\theta \quad (10)$$

This results in the appearance of some additional resonant terms in the slow-flow (9):

$$\frac{dR}{d\eta} - \frac{R}{2} \sin 2\theta = 0 \quad (11)$$

$$R\, \frac{d\theta}{d\eta} - \delta_1\, R - R \cos k\eta - \frac{R}{2} \cos 2\theta = 0 \quad (12)$$

Eq.(11) can be solved in closed form:

$$R = C\, e^{\frac{1}{2}\int \sin 2\theta\, d\eta} \quad (13)$$

where C is an arbitrary constant. Eq.(13) gives R as a function of θ, and θ itself is determined by eq.(12), i.e. by

$$\frac{d\theta}{d\eta} = \delta_1 + \frac{1}{2} \cos 2\theta + \cos k\eta \quad (14)$$

Although the R and θ variables are therefore uncoupled, the question of the boundedness of x_0 is governed by the behavior of $R(\eta)$, cf. Eq. (7). We are therefore interested in how the behavior of θ, determined by Eq. (14), influences the boundedness of R via Eq. (13).

The answer is this: If Eq. (14) exhibits a limit cycle on the $\theta - \eta$ phase torus, then R is an exponential function of η and the x_0 motion is unstable. This is because if $\theta(\eta)$ is a periodic function mod $n\pi$, then so is $\sin 2\theta$, but with non-zero average value taken over one orbit of the limit cycle (in general). Thus the integral in Eq. (13), $\int \sin 2\theta\, d\eta$, taken over one cycle, will not in general be zero, and R will grow or decay exponentially in η. Since the Wronskian of Eq. (2) is constant in t,

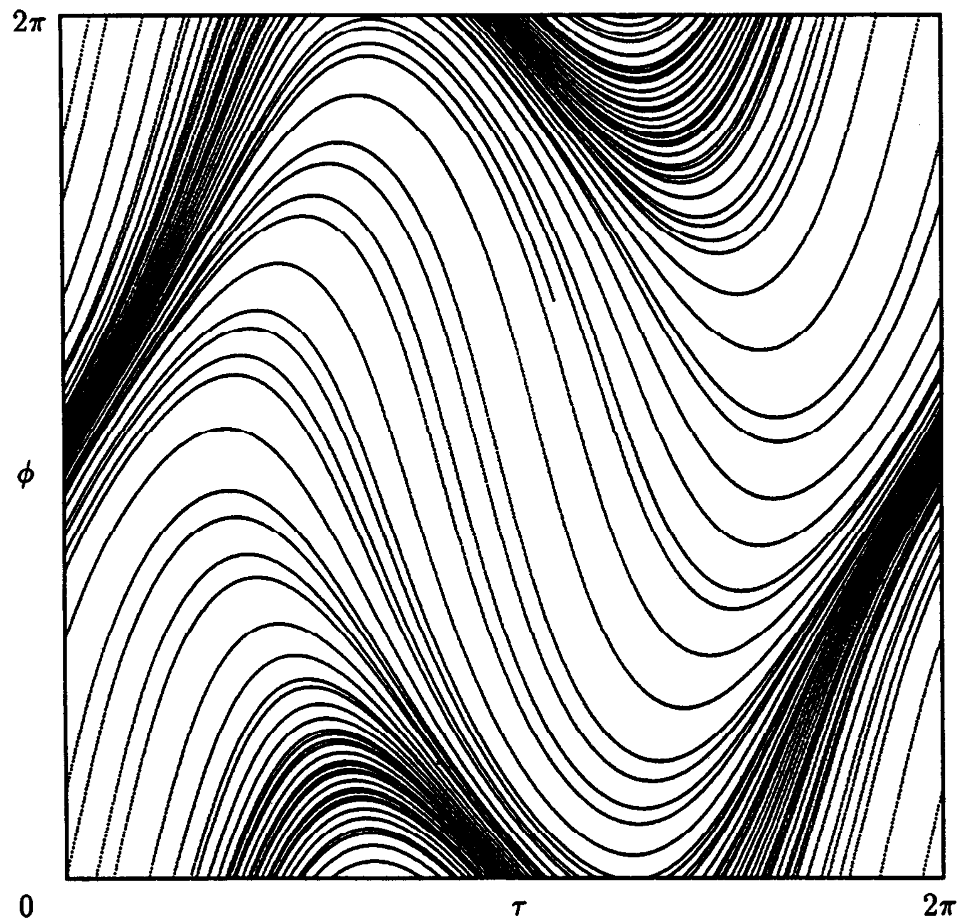

Fig.6. Numerical integration of Eq. (15) for $\delta_1 = 0.1$ and $k = 0.7$ for the initial condition $\phi = 0, \tau = 0$.

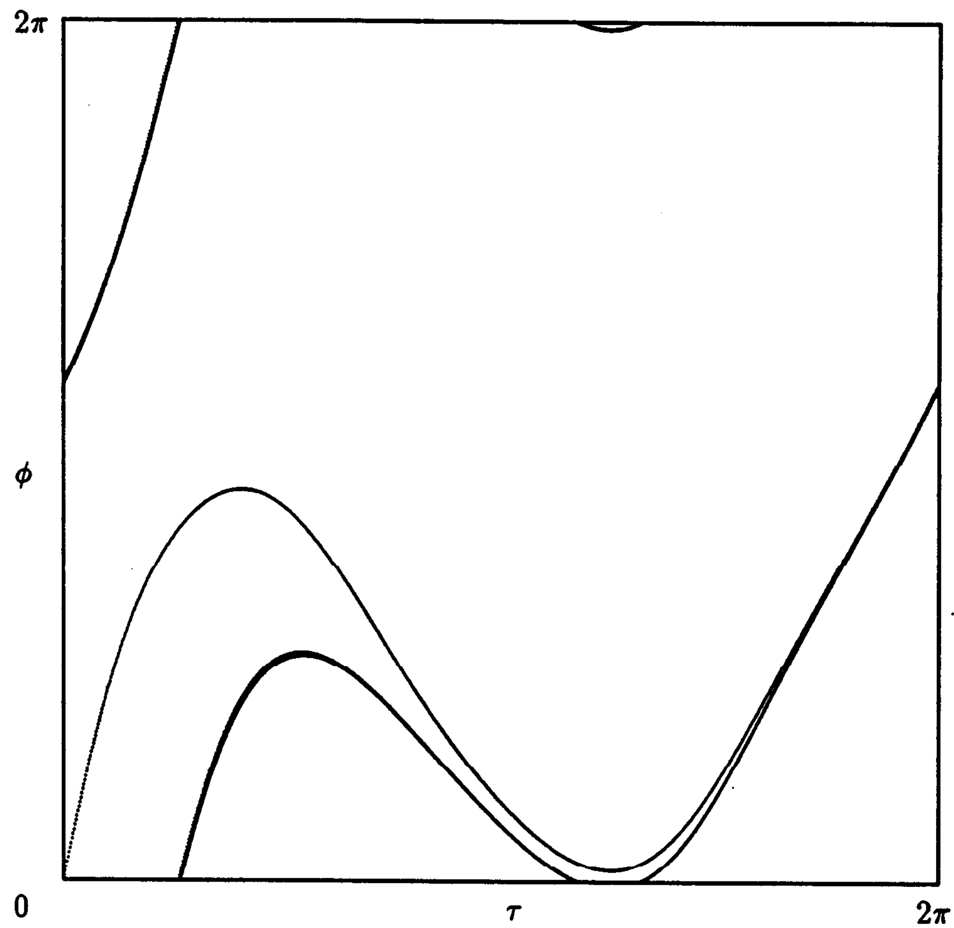

Fig.7. Numerical integration of Eq. (15) for $\delta_1 = 0.2$ and $k = 0.7$ for the initial condition $\phi = 0, \tau = 0$.

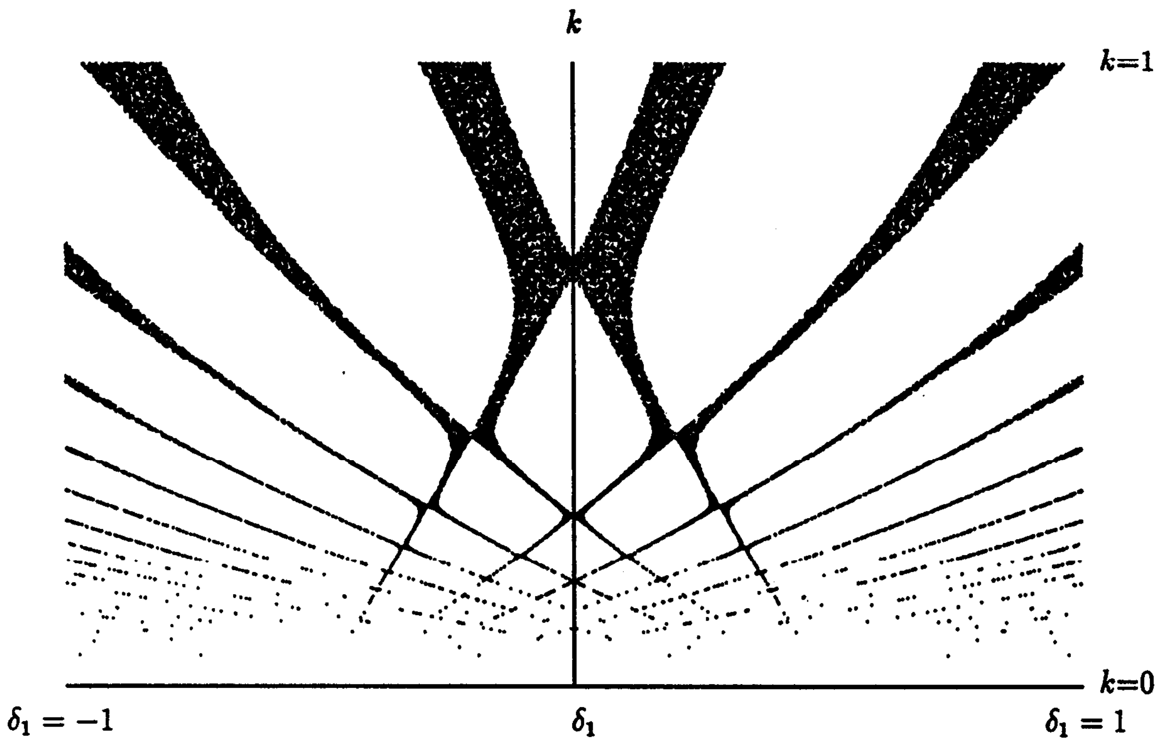

Fig.8. Stability of Eq. (2) as obtained by perturbation method. Eq. (15) was numerically integrated for thousands of points randomly chosen in the (δ_1, k) parameter plane. Black=no limit cycle=stable, white=limit cycle=unstable. Compare with Fig.9 where $\delta = \frac{1}{4} + \epsilon\delta_1 + \cdots$, $\omega = k\epsilon$.

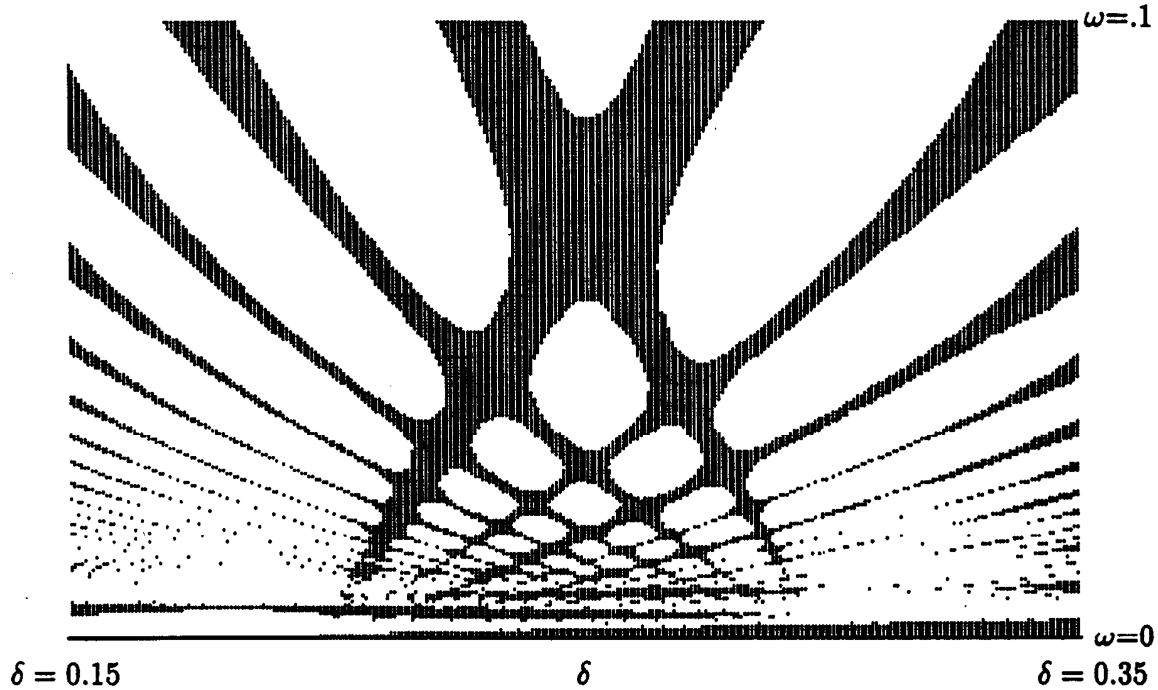

Fig.9. Stability of Eq. (2) for $\epsilon = 0.1$ as obtained by direct numerical integration. This Figure is an enlargement of a portion of Fig.3. Black=stable, white=unstable. Compare with Fig.8 where $\delta = \frac{1}{4} + \epsilon\delta_1 + \cdots$, $\omega = k\epsilon$.

an exponentially decaying solution must be accompanied by a second linearly independent solution which is exponentially growing. Thus a limit cycle in Eq. (14) corresponds to the occurrence of an unbounded solution in Eq. (2) . If, on the other hand, the torus flow (14) is equivalent to an irrational flow, then the integral $\int sin 2\theta \, d\eta$ will tend to zero on the average, R will remain bounded as $t \rightarrow \infty$, and the x_0 motion will be stable.

So the question of the stability of the QP Mathieu equation (2) is reduced to the question of whether Eq. (14) has a limit cycle. Unfortunately, a closed form solution of Eq. (14) is unavailable. Nevertheless, we may numerically integrate Eq. (14) for given values of the parameters δ_1 and k, and determine whether the phase flow exhibits a limit cycle by inspection. Before doing so, we change variables so that the phase torus is $2\pi \times 2\pi$: we set $\phi = 2\theta$ and $\tau = k\eta$, giving

$$\frac{d\phi}{d\tau} = \frac{2\delta_1 + cos\phi + 2cos\tau}{k} \tag{15}$$

Figs.6 and 7 display the phase torus of Eq. (15) for parameters $(\delta_1, k) = (0.1, 0.7)$ and $(0.2, 0.7)$, respectively. Note that Fig.6 displays a flow which appears to be equivalent to an irrational flow, while Fig.7 exhibits a stable limit cycle.

The process of deciding whether a given phase portrait exhibits a limit cycle can be automated by numerically generating a Poincare map corresponding to the surface of section $\tau = \pi$. A stable limit cycle then corresponds to a stable fixed point of the associated one-dimensional circle map. Fig.8 shows the result of such a procedure, in which thousands of points were chosen at random in the (δ_1, k) parameter space, and the presence of a dot represents stability, i.e., the absence of a stable limit cycle.

Fig.8 is symmetric about the k-axis. This may be shown by defining

$$\tilde{\phi} = -(\phi + \pi), \quad \tilde{\tau} = \tau + \pi, \quad \tilde{\delta_1} = -\delta_1 \tag{16}$$

Eq. (15) is unchanged by the transformation (16):

$$\frac{d\tilde{\phi}}{d\tilde{\tau}} = \frac{2\tilde{\delta_1} + cos\tilde{\phi} + 2cos\tilde{\tau}}{k} \tag{17}$$

But since the transformation (16) is a rigid motion (a translation and a reflection), it can't change the phase portrait. Thus the stability is unchanged when δ_1 is replaced by $-\delta_1$.

CONCLUSION

We have presented two schemes for determining the stability of the QP Mathieu equation(2). The first involves numerically integrating Eq. (2), and using a practical criterion for approximately determining whether a given motion is growing unbounded, see Figs.1-5. The second involves a perturbation method which is valid for small ω, in the neighborhood of $\delta = \frac{1}{4}$. The criterion for stability was the presence of an irrational torus flow in Eq. (15), see Fig.8.

These two approaches can be compared by enlarging the region around $\delta = \frac{1}{4}$ in Fig.3 for which $\epsilon = 0.1$, see Fig.9. From the perturbation method, $\delta = \frac{1}{4} + \epsilon\delta_1 + \cdots$ and $\omega = k\epsilon$. We note that there is a strong resemblance between the perturbation results of Fig.8 and the direct numerical treatment of Fig.9.

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