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ORIGINAL ARTICLE

Trigonometric simplification of a class of conservative nonlinear oscillators

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Abstract This paper deals with a class of conservative nonlinear oscillators of the form $\ddot{x}(t) + f(x(t)) = 0$, where $f(x)$ is analytic. A transformation of time from t to a new time coordinate τ is defined such that periodic solutions can be expressed in the form $x(\tau) = A_0 + A_1 \cos 2\tau$. We refer to this process of trigonometric simplification as *trigonometrification*. Application is given to the stability of nonlinear normal modes (NNMs) in two-degree-of-freedom systems.

Keywords Nonlinear normal modes · Stability · Trigonometrification

1 Introduction

It is well known that the nonlinear oscillator given by the ODE

$$\frac{d^2x}{dt^2} + x + x^3 = 0 \tag{1}$$

has a solution which can be written in terms of the Jacobian elliptic function cn [1, 2]:

$$x(t) = A\ \mathrm{cn}(\alpha t, k) \tag{2}$$

where the constants α and k are related to the amplitude A as follows:

$$\alpha = \sqrt{1 + A^2}, \qquad k = \frac{A}{\sqrt{2(1 + A^2)}} \tag{3}$$

It is also well known that a transformation of time from t to τ permits the solution (2) to be written in a simplified form, namely [3]

$$x(\tau) = A \cos \tau \tag{4}$$

where t and τ are related by the Equation [4]

$$dt = \frac{d\tau}{\alpha\sqrt{1 - k^2 \sin^2 \tau}} \tag{5}$$

For applications which involve manipulations of the solution to Equation (1), it is naturally more convenient to use the form (4) than the form (2). As an example, consider the question of the stability of a nonlinear normal mode (NNM) in a two-degree-of-freedom system which is defined by the following expressions for kinetic T and potential V energies [3]:

$$T = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 \tag{6}$$

$$V = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 \tag{7}$$

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Lagrange's equations for this system are

$$\ddot{x} + x + x^3 + xy^2 = 0 \tag{8}$$

$$\ddot{y} + y + x^2 y = 0 \tag{9}$$

where dots represent differentiation with respect to t. This system exhibits the exact solution (the x-mode)

$$x = A\,\mathrm{cn}(\alpha t, k), \quad y = 0 \tag{10}$$

where α and k are given by Equation (3). To investigate the stability of this mode, we set

$$\begin{aligned} x &= A\,\mathrm{cn}(\alpha t, k) + u(t) \\ y &= v(t) \end{aligned} \tag{11}$$

Substituting (11) into (8), (9) and linearizing in $u(t)$ and $v(t)$ results in

$$\begin{aligned} &\ddot{u} + u + 3A^2\,\mathrm{cn}^2(\alpha t, k)\,u = 0 \\ &\ddot{v} + v + A^2\mathrm{cn}^2(\alpha t, k)\,v = 0 \end{aligned} \tag{12}$$

The first of Equation (12) determines the stability of the motion (10) in the invariant manifold $y = 0$, that is, in the x–$\dot{x}$ phase plane. This is well known to be Liapunov unstable due to phase shear, that is, due to the change in period associated with a change in amplitude, but is orbitally stable [5]. This effect is well understood and is of no interest to us here.

We are rather interested in the boundedness of solutions to the second of Equation (12), the v-equation, which determines the stability of the invariant manifold $y = 0$. The NNM (10) will be said to be stable if all solutions of the v-equation are bounded, and unstable if an unbounded solution exists.

The presence of the elliptic function coefficient in the v-equation makes the analysis of this equation difficult. However, the v-equation can be simplified by using the transformation (5), replacing t by τ as independent variable. This results in the new v-Equation [3]

$$\begin{aligned} &(3A^2 + 4 + A^2\cos 2\tau)\,v'' - A^2 \sin 2\tau\, v' \\ &\quad + (4 + 2A^2 + 2A^2\cos 2\tau)\,v = 0 \end{aligned} \tag{13}$$

where primes denote differentiation with respect to τ. Note that Equation (13) is exact, i.e., no assumption of small amplitude A has been made. The boundedness of solutions in Equation (13) can be investigated by using the method of harmonic balance [3, 6], i.e., by expanding v in a Fourier series.

To summarize, the stability analysis of the NNM (11) has been simplified by using the transformation (5) of time from t to τ, which replaced the elliptic cn function in the v-Equation (12), by trig functions in Equation (13).

In this paper, we generalize this idea, replacing Equation (1) by a conservative nonlinear oscillator equation of the form

$$\frac{d^2x}{dt^2} + f(x) = 0 \tag{14}$$

where $f(x)$ is an analytic function of x. Of course, an equation of the form (14) will not in general have an elliptic integral solution. Nevertheless, we show how to produce a time transformation from t to new time τ which allows the periodic solution of (14) to be expressed in terms of a cosine function. We will refer to this process of trigonometric simplification by the neologism *trigonometrification*.

2 Trigonometrification

In this section, we derive the transformation (5) which trigonometrifies Equation (1) *without* using the fact that the solution to (1) involves the elliptic function cn. The procedure we use here will be shown later in this paper to be applicable to a general class of nonlinear oscillator equations.

Using the form of Equation (5) as a model, we assume a time transformation of the form

$$dt = \frac{d\tau}{\sqrt{g(\tau)}} \tag{15}$$

where $g(\tau)$ is to be found. Using Equation (15) to transform Equation (1) results in

$$x''g + \frac{1}{2}x'g' + x + x^3 = 0 \tag{16}$$

where primes denote derivatives with respect to τ. We can turn this into an equation on g

$$g' + \frac{2x''}{x'}g + \frac{2(x + x^3)}{x'} = 0 \tag{17}$$

We want the time transformation to give us $x(\tau) = A\cos\tau$, so we assume this solution for x. We substitute $x(\tau) = A\cos\tau$ into Equation (17) and obtain a first-order linear ODE on $g(\tau)$

$$g' + \frac{2}{\tan\tau} g + \frac{-2}{A\sin\tau}(A\cos\tau + A^3\cos^3\tau) = 0 \quad (18)$$

The homogeneous part of Equation (18)

$$g' + \frac{2}{\tan\tau} g = 0 \quad (19)$$

has the solution

$$g(\tau) = \frac{K}{\sin^2\tau} \quad (20)$$

where K is an arbitrary constant. Using variation of parameters, we seek a solution to Equation (18) in the form

$$g(\tau) = \frac{K(\tau)}{\sin^2\tau} \quad (21)$$

Plugging (21) into Equation (18) and solving for $K'(\tau)$ yields

$$K'(\tau) = 2\sin\tau(\cos\tau + A^2\cos^3\tau) \quad (22)$$

Integrating, we obtain

$$K(\tau) = \int 2\sin\tau(\cos\tau + A^2\cos^3\tau)\, d\tau \quad (23)$$

We solve the integral using the substitution of $u = \cos\tau$ and find

$$K(\tau) = -\left(\cos^2\tau + \frac{1}{2}A^2\cos^4\tau\right) + C \quad (24)$$

where C is an arbitrary constant. This gives $g(\tau)$ in the form

$$g(\tau) = \frac{-1}{\sin^2\tau}\left(\cos^2\tau + \frac{1}{2}A^2\cos^4\tau - C\right) \quad (25)$$

We note that $g(\tau)$ has singularities at $\tau = 0$ and π. These singularities are undesirable, so we choose C appropriately to remove them. To do this, we let $\cos^2\tau = 1 - \sin^2\tau$ and simplify.

$$g(\tau) = \frac{-(1 + (1/2)A^2 - C)}{\sin^2\tau} + (1 + A^2) - \frac{1}{2}A^2\sin^2\tau \quad (26)$$

Setting $C = 1 + (1/2)A^2$ removes the singularities at $\tau = 0$ and π and we are left with

$$g(\tau) = (1 + A^2) - \frac{1}{2}A^2\sin^2\tau \quad (27)$$

Substituting this back into our original ansatz (15), we find

$$dt = \frac{d\tau}{\sqrt{(1 + A^2) - \frac{1}{2}A^2\sin^2\tau}} \quad (28)$$

Using the expressions for α and k given in Equation (3), we obtain

$$dt = \frac{d\tau}{\alpha\sqrt{(1 - k^2\sin^2\tau)}} \quad (29)$$

which is the same as Equation (5).

3 Generalization

In this section, we generalize the trigonometrification process to apply to equations of the form

$$\ddot{x} + f(x) = 0 \quad (30)$$

where we assume f is odd, $f(-x) = -f(x)$. We seek to stretch the time in Equation (30) so that the transformed equation has the solution $x(\tau) = A\cos(\tau)$. As in the previous section, we assume a time transformation of the form

$$dt = \frac{d\tau}{\sqrt{g(\tau)}} \quad (31)$$

where $g(\tau)$ is to be found. Equation (31) turns Equation (30) into

$$x''g + \frac{1}{2}x'g' + f(x) = 0 \quad (32)$$

We want $x(\tau)$ to have a solution in the form $x(\tau) = A\cos\tau$. Thus, plugging $x(\tau) = A\cos\tau$ into Equation (32) yields

$$g' + \frac{2}{\tan\tau}g + \frac{-2}{A\sin\tau}f(A\cos\tau) = 0 \tag{33}$$

As in the previous section, we look for a solution to Equation (33) in the form of Equation (21)

$$g(\tau) = \frac{K(\tau)}{\sin^2\tau} \tag{34}$$

Plugging this into Equation (33) and solving for $K'(\tau)$ we find

$$K'(\tau) = \frac{2}{A}\sin\tau\, f(A\cos\tau) \tag{35}$$

Integrating, we obtain

$$K(\tau) = \int \frac{2}{A}\sin\tau\, f(A\cos\tau)d\tau \tag{36}$$

We evaluate this integral by using the trig substitution $u = \cos\tau$ and find

$$K(\tau) = -\frac{2}{A^2}F(A\cos\tau) + C \tag{37}$$

where F is defined by $F'(x) = f(x)$. Our equation for g, Equation (34), then becomes

$$g(\tau) = \frac{1}{\sin^2\tau}\left(-\frac{2}{A^2}F(A\cos\tau) + C\right) \tag{38}$$

We wish to choose C such that $g(\tau)$ has no singularities at $\tau = 0$ or π. We note that

$$F(A\cos\tau)|_{\tau=0} = F(A) \quad \text{and}$$
$$F(A\cos\tau)|_{\tau=\pi} = F(-A) \tag{39}$$

Our assumption that $f(x)$ is odd means $F(A)$ is even, thus $F(A) = F(-A)$. We thus choose $C = 2F(A)/A^2$ to remove the singularities. The expression for the time transformation becomes

$$g(\tau) = \frac{-2}{A^2\sin^2\tau}\left(F(A\cos\tau) - F(A)\right) \tag{40}$$

4 Example 1

As an example of the application of the previous formula (40), we consider the following system, which has no known closed form solution:

$$\ddot{x} + x + x^5 = 0 \tag{41}$$

We begin by computing $F(x)$ as the antiderivative of $f(x)$=$x + x^5$

$$F(x) = \frac{x^2}{2} + \frac{x^6}{6} \tag{42}$$

Substituting Equation (42) into Equation (40) gives the following expression for $g(\tau)$:

$$g(\tau) = \frac{-2}{A^2\sin^2\tau}\left(\frac{1}{2}A^2\cos^2\tau + \frac{1}{6}A^6\cos^6\tau \right.$$
$$\left. - \left(\frac{1}{2}A^2 + \frac{1}{6}A^6\right)\right) \tag{43}$$

which reduces to

$$g(\tau) = 1 + A^4\left(1 - \sin^2\tau + \frac{1}{3}\sin^4\tau\right) \tag{44}$$

resulting in the time transformation

$$dt = \frac{d\tau}{\sqrt{1 + A^4\left(1 - \sin^2\tau + \frac{1}{3}\sin^4\tau\right)}} \tag{45}$$

As a check, the transformation (45) applied to Equation (41) gives

$$g(\tau)x'' + \frac{1}{2}g'(\tau)x' + x + x^5 = 0 \tag{46}$$

which becomes, using Equation (44),

$$\left(1 + A^4\left(1 - \sin^2\tau + \frac{1}{3}\sin^4\tau\right)\right)x'' + \frac{1}{2}A^4\cos\tau$$
$$\times\left(-2\sin\tau + \frac{4}{3}\sin^3\tau\right)x' + x + x^5 = 0 \tag{47}$$

which turns out to have the exact solution $x(\tau) = A\cos\tau$ as desired.

5 Example 2

In this section, we consider an example for which $f(x)$ in Equation (14) is not a polynomial. We select the familiar example of the pendulum

$$\ddot{x} + \sin x = 0 \tag{48}$$

In this case, $f(x) = \sin x$ giving that $F(x) = -\cos x$. From (40), the associated expression for $g(\tau)$ becomes

$$g(\tau) = \frac{-2}{A^2 \sin^2 \tau} \left(\cos A - \cos\left(A \cos \tau\right)\right) \tag{49}$$

which has the limit of $\sin(A)/A$ as τ goes to 0 or π.

The resulting time transformation is

$$dt = \frac{d\tau}{\sqrt{\frac{-2}{A^2 \sin^2 \tau} \left(\cos A - \cos\left(A \cos \tau\right)\right)}} \tag{50}$$

Thus, the trigonometrified version of the pendulum Equation (48) has the exact solution $x(\tau) = A \cos \tau$

$$g(\tau)x'' + \frac{1}{2}g'(\tau)x' + \sin x = 0 \tag{51}$$

6 What if $f(x)$ is not an odd function? Trigonometrification revisited

So far we have considered only functions $f(x)$ that are odd. In this section, we generalize the trigonometrification process to include a more general class of oscillator equations. We again start with the form

$$\ddot{x} + f(x) = 0 \tag{52}$$

but no longer assume that f is odd. We do however assume that the system (52) exhibits an oscillating solution.

We seek to stretch the time in Equation (52) so that $x(\tau) = Q(\tau)$ where $Q(\tau)$ is periodic, the specific form of $Q(\tau)$ to be determined. We again assume that the time transformation takes the general form

$$dt = \frac{d\tau}{\sqrt{g(\tau)}} \tag{53}$$

The transformation (53) turns Equation (52) into

$$x''g + \frac{1}{2}x'g' + f(x) = 0 \tag{54}$$

Substituting $x(\tau) = Q(\tau)$ into Equation (54) yields

$$g' + 2\frac{Q''}{Q'}g + 2\frac{f(Q)}{Q'} = 0 \tag{55}$$

We assume the solution to Equation (55) is of the form

$$g(\tau) = \frac{K(\tau)}{Q'^2} \tag{56}$$

Plugging this into Equation (55) and solving for $K'(\tau)$ we find

$$K'(\tau) = -2f(Q)Q' \tag{57}$$

Integrating,

$$K(\tau) = -2F(Q) + C \tag{58}$$

where $F'(Q) = f(Q)$, i.e., $F(Q)$ is the antiderivative of $f(Q)$. Our equation for g, Equation (56), then becomes

$$g(\tau) = \frac{-2F(Q) + C}{Q'^2} \tag{59}$$

We wish to choose C such that there are no singularities at τ^* (where τ^* is defined such that $Q'(\tau^*) = 0$). Thus, we choose $C = 2\, F(Q)|_{\tau=\tau^*}$.

$$g(\tau) = 2\,\frac{F(Q)|_{\tau=\tau^*} - F(Q)}{Q'^2} \tag{60}$$

Note that the more complicated $Q(\tau)$ becomes, the more τ^* exist and the harder it will be to remove the singularities for all τ^*. However, it can be shown [7] that the ansatz

$$Q(\tau) = A_0 + A_1 \cos 2\tau \tag{61}$$

is sufficient to treat systems for which $f(x)$ is an arbitrary polynomial. Assuming the form (61) for $Q(\tau)$, we find

$$g(\tau) = 2\,\frac{F(A_0 + A_1 \cos 2\tau^*) - F(A_0 + A_1 \cos 2\tau)}{4A_1^2 \sin^2 2\tau} \tag{62}$$

Singularities exist at $\tau^* = 0$ and at $\tau^* = \pi/2$. If we choose $C = 2\,F(Q)|_{\tau=0} = F(A_0 + A_1)$, we remove the singularity at $\tau^* = 0$. To remove the singularity at $\tau^* = \pi/2$, we must also determine an appropriate relationship between A_0 and A_1. To do this, we expand the numerator of $g(\tau)$ in Equation (62) in a Fourier series and convert all even powers of $\cos 2\tau$ to even powers of $\sin 2\tau$ via the identity $\cos^2 2\tau = 1 - \sin^2 2\tau$. This results in the following expression for the numerator of $g(\tau)$:

$$\begin{aligned}\text{numerator}(g(\tau)) = q_0 + p_0 \cos 2\tau + \sin^2 2\tau \\ \times\ (q_1 + p_1 \cos 2\tau) + \cdots \\ +\ \sin^{2n} 2\tau\,(q_n + p_n \cos 2\tau)\end{aligned} \tag{63}$$

where $q_n = q_n(A_0, A_1)$ and $p_n = p_n(A_0, A_1)$. The $\sin^{2n} 2\tau$ in front of $q_n + p_n \cos 2\tau$ $(n \geq 1)$ eliminates any possible singularities coming from these terms. Moreover, our choice of $C = F(A_0 + A_1)$ removed the singularity at $\tau^* = 0$, which requires that $q_0 + p_0 = 0$, i.e., $q_0 = -p_0$. It remains to remove the singularity at $\tau^* = \pi/2$, which requires that $q_0 - p_0 = -2p_0 = 0$. Finally, p_0 is made to vanish by choosing an appropriate relationship between A_0 and A_1. It turns out that the resulting equation $p_0(A_0, A_1) = 0$ is an $(n+1)$th degree polynomial equation where n is the polynomial degree of $f(x)$. The procedure is illustrated by the following example.

7 Example 3

We take as an example the strongly nonlinear system [8]

$$\ddot{x} + x^3 + x^4 = 0 \tag{64}$$

Here $f(x) = x^3 + x^4$ which gives the antiderivative $F(x) = \frac{1}{4}x^4 + \frac{1}{5}x^5$. Assuming a trigonometrified solution, Equation (61), and substituting into Equation (62) results in a time transformation of the form $dt = d\tau/\sqrt{g(\tau)}$ where

$$g(\tau) = \frac{\frac{1}{4}(A_0 + A_1)^4 + \frac{1}{5}(A_0 + A_1)^5 - \frac{1}{4}(A_0 + A_1\cos 2\tau)^4 - \frac{1}{5}(A_0 + A_1 \cos 2\tau)^5}{2A_1^2 \sin^2 2\tau} \tag{65}$$

which simplifies to the form

$$\begin{aligned}g(\tau) = \frac{1}{2A_1^2 \sin^2 2\tau}\big(q_0 + p_0 \cos 2\tau + (q_1 + p_1 \cos 2\tau) \\ \times \sin^2 2\tau + (q_2 + p_2 \cos 2\tau)\sin^4 2\tau\big)\end{aligned} \tag{66}$$

where

$$\begin{aligned}
q_0 &= A_0^4 A_1 + A_0^3 A_1 + 2A_0^2 A_1^3 + A_0 A_1^3 + \tfrac{1}{5}A_1^5 = -p_0 \\
p_0 &= -A_0^4 A_1 - A_0^3 A_1 - 2A_0^2 A_1^3 - A_0 A_1^3 - \tfrac{1}{5}A_1^5 \\
q_1 &= 2A_0^3 A_1^2 + \tfrac{3}{2}A_0^2 A_1^2 + 2A_0 A_1^4 + \tfrac{1}{2}A_1^4 \\
p_1 &= A_0 A_1^3 + 2A_0^2 A_1^3 + \tfrac{2}{5}A_1^5 \\
q_2 &= -A_0 A_1^4 - \tfrac{1}{4}A_1^4 \\
p_2 &= -\tfrac{1}{5}A_1^5
\end{aligned} \tag{67}$$

As stated in the previous section, the choice of $C = F(A_0 + A_1)$ is responsible for $q_0 = -p_0$, cf. Equation (67). Thus, setting $p_0 = 0$ will define a relationship between A_0 and A_1 that eliminates the singularities in $g(\tau)$. With this in mind we set $q_0 = -p_0 = 0$ to find

$$A_0^4 A_1 + A_0^3 A_1 + 2A_0^2 A_1^3 + A_0 A_1^3 + \frac{1}{5}A_1^5 = 0 \tag{68}$$

Note that this is a fifth degree polynomial equation, which is one degree higher than that of $f(x) = x^3 + x^4$, as stated at the end of the previous section. Here the relationship between A_0 and A_1 produces real solutions for a certain set of A_0 and A_1 values. Assuming a real solution, we obtain the following final expression for $g(\tau)$:

$$g(\tau) = \frac{1}{2A_1^2}\big(q_1 + p_1 \cos 2\tau + (q_2 + p_2 \cos 2\tau)\sin^2 2\tau\big) \tag{69}$$

which becomes

$$
\begin{aligned}
g(\tau) = {} & A_0^3 + \frac{3}{4}A_0^2 + A_0 A_1^2 + \frac{1}{4}A_1^2 \\
& + \left(\frac{1}{2}A_0 A_1 + A_0^2 A_1 + \frac{1}{5}A_1^3\right)\cos 2\tau \\
& - \left(\frac{1}{2}A_0 A_1^2 + \frac{1}{8}A_1^2\right)\sin^2 2\tau \\
& - \frac{1}{10}A_1^3 \cos 2\tau \sin^2 2\tau \qquad (70)
\end{aligned}
$$

In order to visualize the process of trigonometrification, we show in Fig.1 the periodic solution to Equation (64) for the initial condition $x(0) = 0.6058$, $\dot{x}(0) = 0$. Then in Fig.2 we show the trigonometrified solution, which is of the form $x(\tau) = A_0 + A_1 \cos 2\tau$, where $A_0 = -0.1948$ and $A_1 = 0.8006$. These values for A_0 and A_1 are obtained by simultaneously solving the initial condition $A_0 + A_1 = x(0)$ together with Equation (68). These two figures also show the relative time compression involved in the trigonometrification process.Figure 3 shows the corresponding relationship between the original time t and transformed time τ defined by $dt = d\tau/\sqrt{g(\tau)}$ where $g(\tau)$ is given by Equation (70).

Figure 4 compares a variety of solutions to Equation (64) for different initial conditions with their trignometrified counterparts. Note that the level curves of the original system are particularly distorted as they approach a separatrix with a saddle point at $x = -1$, $v = \dot{x} = 0$. Since our method is limited to periodic solutions of the differential equation, we are limited to looking inside the separatrix. This turns out to yield a maximum permissible value for A_1, namely $A_1 = 0.8029$, which corresponds to $A_0 = -0.1971$. Thus, for this problem A_0 ranges from -0.1971 to 0.

As an application of this result, suppose we are interested in the stability of the NNM which lies in the

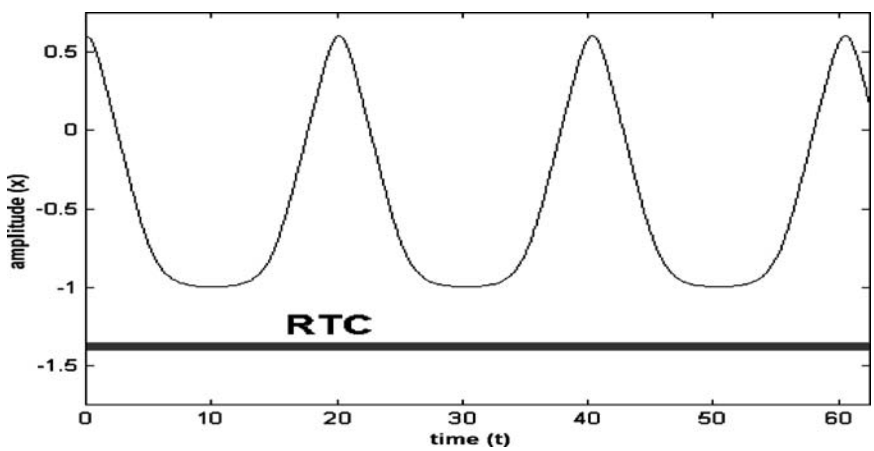

Fig. 1 Periodic solution $x(t)$ to Equation (64) for initial condition $x(0) = 0.6058, \dot{x}(0) = 0$. Result obtained by numerical integration. Note that there is no relative time compression (RTC), corresponding to the original time

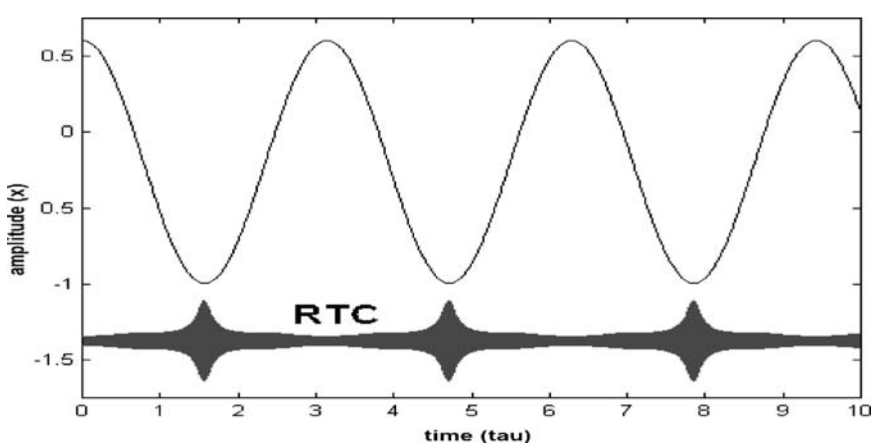

Fig. 2 Trigonometrified solution $x(\tau) = -0.1948 + 0.8006 \cos 2\tau$ corresponding to original periodic solution of Fig.1. Comparison with Fig. 1 shows that the relative time compression (RTC) is greatest where the original periodic motion is stalled, that is, where the plot in Fig. 1 has nearly flat horizontal segments

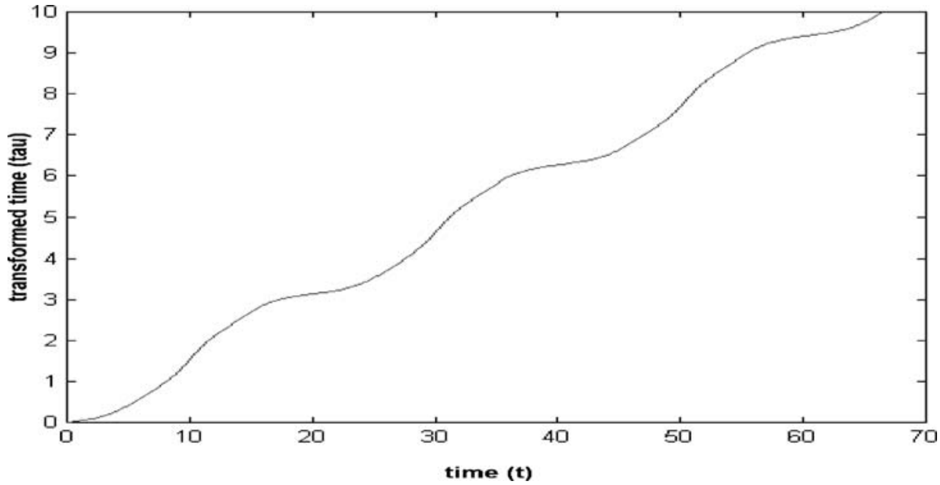

Fig. 3 Transformed time τ shown as a function of original time t. Result obtained by numerical integration of $dt = d\tau/\sqrt{g(\tau)}$ where $g(\tau)$ is given by Equation (70), and where $A_0 = -0.1948$ and $A_1 = 0.8006$

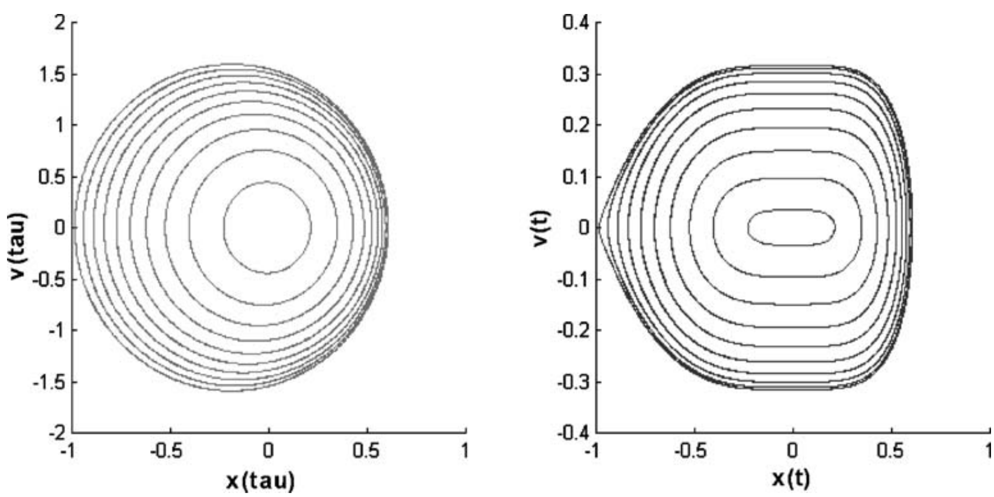

Fig. 4 Phase plane plots of solutions to Equation (64) (*right*) compared to their trigonometrified counterparts (*left*)

$y = 0$ invariant manifold of the following system [8]:

$$\ddot{x} + x^3 + x^4 + xy^2 = 0, \qquad \ddot{y} + \omega^2 y + x^2 y = 0 \tag{71}$$

Note that the absence of a linear term in the equation for the x-mode, Equation (64), makes it difficult to obtain an expression for the NNM, and therefore makes the stability problem difficult without trigonometrification. Using Equation (70) for $g(\tau)$ to define the time transformation results in stability of the x-mode in Equation (71) being governed by

$$h_1(\tau)v'' + h_2(\tau)v' + (\omega^2 + h_3(\tau))v = 0 \tag{72}$$

where

$$\begin{aligned} h_1 &= -\tfrac{1}{8A_1^2}\big(p_2\cos 6\tau + 2q_2\cos 4\tau - p_2\cos 2\tau \\ &\quad -4p_1\cos 2\tau - 2q_2 - 4q_1\big) \\ h_2 &= \tfrac{1}{8A_1^2}(3p_2\sin 6\tau + 4q_2\sin 4\tau - p_2\sin 2\tau \\ &\quad -4p_1\sin 2\tau) \\ h_3 &= \tfrac{1}{2}A_1^2\cos 4\tau + A_0A_1\cos 2\tau + \tfrac{1}{2}A_1^2 + A_0 \end{aligned} \tag{73}$$

Equation (72) is a generalized Ince's Equation [3] and can be investigated by using harmonic balance.

8 Conclusions

We have presented a scheme for reparametrizing time such that the periodic motion of a general class of conservative nonlinear oscillators is able to be represented by a simple cosine function. Specifically, if the oscillator is of the form

$$\frac{d^2x}{dt^2} + f(x) = 0 \tag{74}$$

then, when expressed in the new time τ, the periodic motion may be written in the form

$$x(\tau) = A_0 + A_1 \cos 2\tau \quad \text{for general } f(x), \quad \text{and} \tag{75}$$

$$x(\tau) = A \cos \tau \quad \text{for } f(x) \text{ odd, i.e., } f(-x) = -f(x) \tag{76}$$

We have shown that this procedure has application to the stability of NNMs in two-degree-of-freedom systems. Specifically, the stability problem is reduced to the study of a linear ODE with trigonometric coefficients. See, e.g., Equations (13) and (72). Note that the reason this works is because the question of stability is invariant under reparametrization in time. Other applications, not covered in this paper, would include bifurcation of periodic orbits resulting from changes in stability. In the case of conservative two-degree-of-freedom systems like that of Equations (6)–(9), or of Equation (71), this would involve trigonometrification of both nonlinear equations.

We note that although the process of trigonometrification has the obvious advantage of replacing the original time dependence of the periodic motion in question with a trigonometrically simplified representation, it does so at the cost of (a) including a first-derivative term in an ODE that originally had none and (b) including time-dependent terms in an ODE which was originally autonomous. As an example of this, see Example 1 where the original ODE, Equation (41), is replaced by the trigonometrified ODE, Equation (54).

Finally, we note that although trigonometrification totally simplifies a particular periodic solution of the original ODE (74), expressing it in one of the forms (75) or (76), it does not simplify the general solution of the original ODE.

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