ON THE STABILITY OF THE VIBRATIONS OF A PARTICLE IN THE PLANE RESTRAINED BY TWO NON-IDENTICAL SPRINGS

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Abstract—The stability of the vibrations of a particle constrained to a plane and restrained by two non-identical springs with initial stress is studied by using Floquet theory and perturbations. It is shown that unexpected regions of instability occur due to the non-identical properties of the springs.

INTRODUCTION

THE stability of the vibrations of a particle constrained to a plane and restrained by two identical springs with initial stress has been studied by Yang and Rosenberg [1, 2], and by Rand and Tseng [3].

In any real system, the springs cannot be made identical. It is therefore of interest to extend the results of [1–3] to include the effects of non-identical springs, especially for springs which are nearly identical.

In this paper it is shown that the nature of the stability diagram is qualitatively changed if the springs are not identical. Unexpected regions of instability occur even if the springs are nearly identical.

FORMULATION

Consider a particle with coordinates (x, y) constrained to the x–y plane and restrained by two non-identical linear springs with initial stress, as in Fig. 1.
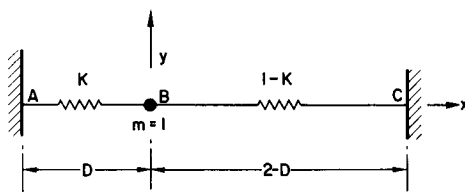

FIG. 1. A particle in the plane.

Let the equilibrium position of the particle be the origin of the x–y coordinate system, oriented such that the larger of the two lengths AB, BC lies in the positive x direction. Choose the unit of mass such that the mass of the particle is unity, the unit of length such that the distance between the supports, AC, is twice unity, and the unit of time such that the sum of the two spring constants is unity.

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Let D be the smaller of the two lengths AB, BC. Then

$$0 < D \leqslant 1 \tag{1}$$

Let K be the spring constant of the spring associated with D. Then

$$0 < K < 1 \tag{2}$$

Let L be the original length of the spring associated with D and K. Then

$$L > 0. \tag{3}$$

Let λ be the original length of the other spring. Since the particle is in equilibrium when $x \equiv y \equiv 0$,

$$K(D - L) = (1 - K)(2 - D - \lambda) \tag{4}$$

Requiring $\lambda > 0$ imposes the condition

$$\lambda = \frac{K(L - D)}{1 - K} + 2 - D > 0 \tag{5}$$

Clearly D, K and L are independent parameters which completely specify the physical properties of the model. For given D and K, from (3) and (5),

$$L > \max\left\{0, D - \frac{(2 - D)(1 - K)}{K}\right\} \tag{6}$$

The equations of motion for the particle are

$$\ddot{x} = (1 - K)\frac{(d_2 - \lambda)}{d_2}(2 - D - x) - K\frac{(d_1 - L)}{d_1}(D + x)$$

$$\ddot{y} = -(1 - K)\frac{(d_2 - \lambda)}{d_2}y - K\frac{(d_1 - L)}{d_1}y$$

where

$$d_1^2 = (D + x)^2 + y^2$$

$$d_2^2 = (2 - D - x)^2 + y^2$$

and where dots represent differentiation with respect to time, t.
    A possible mode of vibration, the x-mode, is

$$x = e \cos t$$

$$y = 0$$

where, in order to avoid collisions of the particle with the support,

$$0 \leqslant e < D \tag{7}$$

The first variational equations corresponding to the x-mode are

$$\ddot{\delta x} + \delta x = 0 \tag{8}$$

$$\ddot{\delta y} + f(t)\,\delta y = 0 \tag{9}$$

where

$$f(t) = K\,\frac{(D - L) + e\cos t}{D + e\cos t} + (1 - K)\,\frac{(2 - D - \lambda) - e\cos t}{(2 - D) - e\cos t} \tag{10}$$

The x-mode is stable only if both (8) and (9) are stable. (An equation is said to be stable if all of its solutions are bounded for all $t > 0$, and unstable if an unbounded solution exists.) Since (8) is stable, the stability of the x-mode is dependent only on the stability of (9).

STABILITY ANALYSIS

A particular case of Hill's equation, (9) may be rewritten as

$$\ddot{z} + f(t)\,z = 0 \tag{11}$$

where

$$z = \delta y$$

$$f(t) = \frac{Ee^2\cos^2 t + Fe\cos t + \Delta}{Ee^2\cos^2 t + Ge\cos t + 1} \tag{12}$$

$$E = -1/[D(2 - D)] \tag{13}$$

$$F = [2K - D]/[D(2 - D)] \tag{14}$$

$$G = 2(1 - D)/[D(2 - D)] \tag{15}$$

and

$$\Delta = 2K(D - L)/([D(2 - D)]) \tag{16}$$

The stability of (11) is dependent upon the values of D, K, L and e. The region of physical interest for each is given by (1), (2), (6) and (7), respectively. D and K are viewed as fixed while L and e are varied in order to find, for given D and K, those values of L and e which are stable. From this point of view the quantities E, F and G are fixed while Δ is permitted to vary because of its linear dependence on L. From (16) and (4) it is clear that Δ is proportional to the initial stress. In what follows, the stable regions of the Δ–e plane will be found for given D and K. This information can then be used to find stable values of L and e from (16).

Substituting (6) into (16) gives the region of physical interest for Δ,

$$\Delta < \min\left\{\frac{2K}{2 - D},\ \frac{2(1 - K)}{D}\right\} \tag{17}$$

If

$$\varDelta^* = \min \left\{ \frac{2K}{2 - D}, \ \frac{2(1 - K)}{D} \right\}$$

then using (1) and (2) it may be shown that

$$\varDelta^* \leqslant 1$$

and hence

$$\varDelta < \varDelta^* \leqslant 1 \tag{18}$$

Before applying the results of Floquet theory to (11), consider two particular cases of stability:

Equation (11) will always have bounded solutions if $F = G$ and $\varDelta = 1$, whence $f(t) \equiv 1$. However, (18) shows that this case is not within the region of physical interest.

Equation (11) will always have unbounded solutions if the denominator of $f(t)$, in (12), vanishes. This condition requires either

$$\cos t = -D/e$$

or

$$\cos t = (2 - D)/e$$

However, $\cos^2 t \leqslant 1$. Thus in order for the denominator of $f(t)$ to vanish, either

$$e^2 \geqslant D^2$$

or

$$e^2 \geqslant (2 - D)^2$$

From (1) and (7) this case is also not within the region of physical interest.

It is well known from Floquet theory (Stoker [4], p. 201) that corresponding to transition values of $\varDelta$ and e (for given D and K) from stability to instability, there must exist at least one periodic solution to (11) of period Ω or 2Ω, where Ω is the least period of $f(t)$. From (12), $\Omega = 2\pi$. Therefore, in order to obtain all transition values of $\varDelta$ and e, it is sufficient to examine solutions of period $4\pi/N$, all of which have period 4π. (Here and in what follows, $N = 0, 1, 2, \ldots$)

Now for $e = 0$ and $\varDelta > 0$, the solutions to (11) are of the form $\sin(\sqrt{\varDelta})\,t$ and $\cos(\sqrt{\varDelta})\,t$, which have period $2\pi/\sqrt{\varDelta}$. Thus for $e = 0$, transition points can occur if

$$2\pi/(\sqrt{\varDelta}) = 4\pi/N$$

or

$$\varDelta = N^2/4 \tag{19}$$

Note that $N = 0$ corresponds to a constant, which is a solution to (11) when $\varDelta = e = 0$, and which may be thought of as a periodic function of period 4π.

For $e = 0$ and $\varDelta \leqslant 0$, (11) has unbounded solutions, and hence the entire negative $\varDelta$ axis is unstable.

In general two transition curves intersect each of the above transition points on the Δ-axis, the solution along one behaving like $\sin{(Nt/2)}$, the solution along the other like $\cos{(Nt/2)}$ for $e = 0$. ($N = 0$ is an exception, and only a single transition curve intersects the Δ-axis at $\Delta = 0$. On this curve the solution behaves like a constant for $e = 0$.) This is the situation for Mathieu's equation, for example (Stoker [4], p. 205).

However, all such transition curves do not necessarily lie within the region of physical interest. (18) shows that only $N = 0, 1, 2$ need be considered, i.e. transition points on the Δ-axis which are of physical interest can occur at most at $\Delta = 0, \frac{1}{4}, 1$.

To obtain explicit expressions for the transition curves which intersect these points, a perturbation method is used (Stoker [4], p. 209).

Expand

$$z(t) = z_0(t) + z_1(t)\,e + z_2(t)\,e^2 + \ldots \tag{20}$$

$$\Delta = N^2/4 + \Delta_1 e + \Delta_2 e^2 + \ldots \tag{21}$$

and substitute (20) and (21) into (11). By equating the coefficients of like powers of e, obtain a linear differential equation with constant coefficients on $z_N(t)$. Requiring $z_N(t)$ to be periodic gives a value for Δ_N. For $N = 1, 2$, z_0 is taken as $\sin{(Nt/2)}$, then as $\cos{(Nt/2)}$, since each choice gives a separate transition curve.

In this manner the equations for the following five transition curves were obtained:

$$\Delta = \frac{2K(1 - K)}{D^2(2 - D)^2}\,e^2 + C_1 e^4 + O(e^6) \tag{22}$$

$$\Delta = \frac{1}{4} - \frac{1 + D - 4K}{4D(2 - D)}\,e + C_2 e^2 - C_3 e^3 + C_4 e^4 + O(e^5) \tag{23}$$

$$\Delta = \frac{1}{4} + \frac{1 + D - 4K}{4D(2 - D)}\,e + C_2 e^2 + C_3 e^3 + C_4 e^4 + O(e^5) \tag{24}$$

$$\Delta = 1 - \frac{(2K + D - 2)(2K + 7D - 8)}{12D^2(2 - D)^2}\,e^2 + C_5 e^4 + O(e^6) \tag{25}$$

$$\Delta = 1 + \frac{(2K + D - 2)(10K - 13D + 8)}{12D^2(2 - D)^2}\,e^2 + C_6 e^4 + O(e^6) \tag{26}$$

where $C_1, \ldots, C_6$ are known functions of D and K, and are listed in the Appendix.

DISCUSSION OF THE TRANSITION CURVES

Figures 2, 3 and 4 show the regions of stability of (11) which are of physical interest for $K = 0.25, 0.5$ and 0.75, respectively, and for various values of D, based on (22)–(26).

The problem reduces to the identical spring case if $K = 0.5$ and $D = 1$. In this case $f(t)$ becomes, from (12),

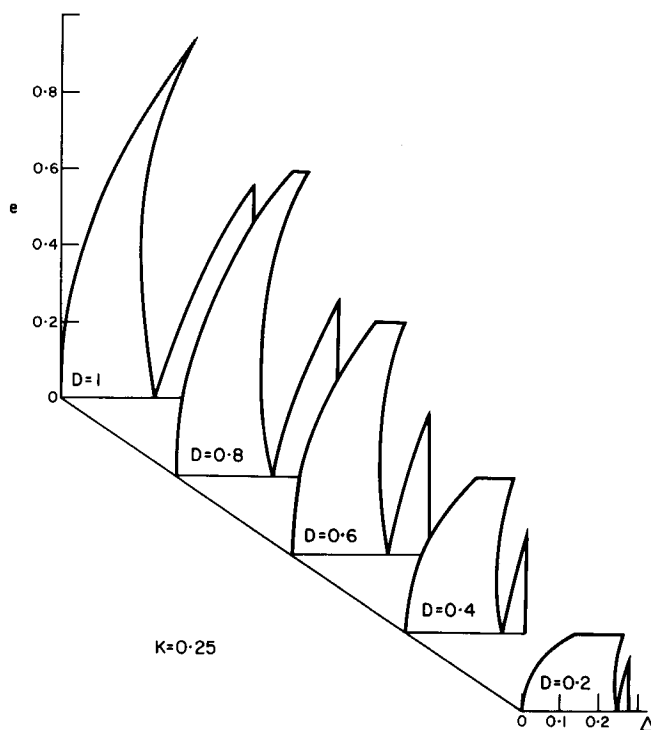$$f(t) = \frac{-\,e^2 \cos^2 t + \Delta}{-\,e^2 \cos^2 t + 1}$$

FIG. 2. Regions of stability of (11) for $K = 0.25$ and for various values of D.

Now the smallest period of $f(t)$ is π, and from the arguments advanced in the previous section, transition points will occur on the Δ-axis at $\Delta = N^2$. Clearly the transition point of the non-identical spring case, $\Delta = \frac{1}{4}$, $e = 0$, is excluded from the identical spring case. Hence as the parameters K and D simultaneously approach 0·5 and 1, respectively, the two transition curves intersecting $\Delta = \frac{1}{4}$, $e = 0$ coalesce (see Fig. 3), and the associated region of instability vanishes.

In addition, it has been shown [3] that in the identical spring case the region of instability associated with $\Delta = 1$, $e = 0$ vanishes. (In fact the region of instability associated with $\Delta = (2N + 1)^2$, $e = 0$ vanishes.)

Hence in the identical spring case the only transition curve which actually separates a region of stability from a region of instability is (22). This system is thus a very special case of the non-identical spring system. For any other values of K and D in the regions (1) and (2), two additional regions of instability may occur, due to transition curves (23)–(26). These regions may or may not lie in the region of physical interest given by (7) and (17).

These additional regions of instability due to (23)–(26) are obviously quite small if the springs are nearly identical. However, there are still other cases for which these regions are small, e.g. if $4K = 1 + D$, then the region of instability associated with (23) and (24) vanishes to $O(e^3)$.
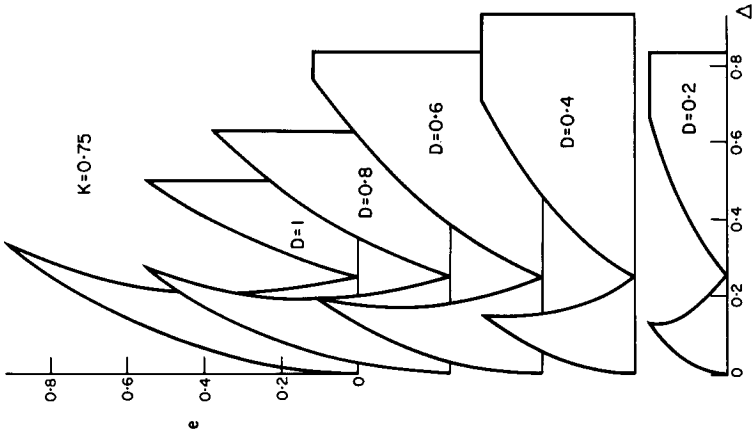
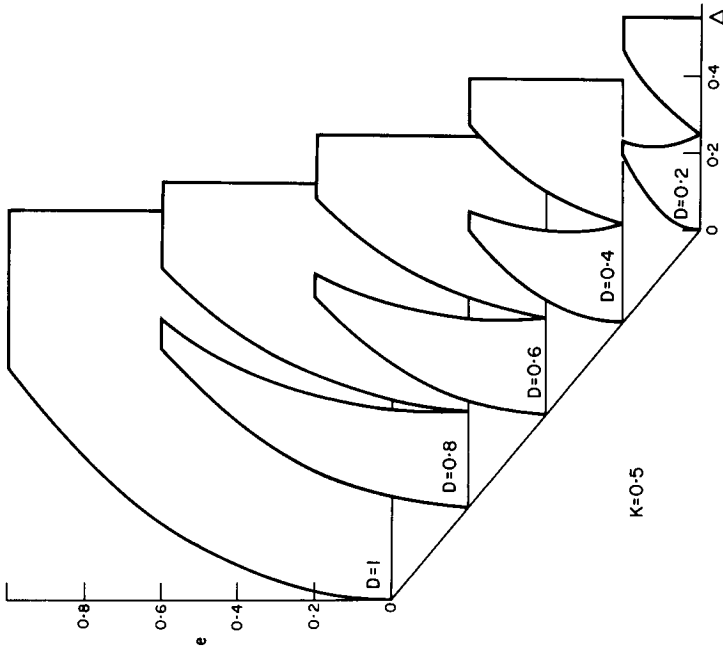FIG. 4. Regions of stability of (11) for $K = 0.75$ and for various values of D.

FIG. 3. Regions of stability of (11) for $K = 0.5$ and for various values of D.

REFERENCES

[1] TA-LUN YANG and R. M. ROSENBERG, On the vibrations of a particle in the plane. *Int. J. non-linear Mech.* **2**, 1–25 (1967).
[2] TA-LUN YANG and R. M. ROSENBERG, On forced vibrations of a particle in the plane. *Int. J. non-linear Mech.* **3**, 47–63 (1968).
[3] RICHARD H. RAND and SHOEI-FU TSENG, On the stability of a differential equation with application to the vibrations of a particle in the plane. To be published in *J. appl. Mech.*
[4] J. J. STOKER, *Nonlinear Vibrations.* Interscience (1950).

APPENDIX

The C_i appearing in (22)–(26) are listed below:

$$32C_1 = (FB_1 + E)\,[3E + B_1(8F - B_2)]$$
$$8C_2 = -3E - B_3B_4$$
$$64C_3 = 5EB_3 + (2B_3^2 - 3E)\,B_4$$
$$32C_4 = B_4[(5E - 8C_2 + 16A_1)\,A_3 - A_2(4F - 25G)] + 2E(5A_1 + 21A_2)$$
$$16C_5 = B_1B_2[E + B_2(B_1/27 - B_5/72)]$$
$$8C_6 = B_1[-F(3E + 4A_4) + B_2(5E/6 - 2A_4/9 - B_2B_5/144)]$$

where

$$32A_1 = 3E - 2B_3^2$$
$$96A_2 = 3E + 2B_3B_4$$
$$4A_3 = B_3$$
$$12A_4 = B_1(5F + 4G)$$
$$B_1 = F - G$$
$$B_2 = F - 4G$$
$$B_3 = (4F - G)/4$$
$$B_4 = (4F - 9G)/4$$
$$B_5 = F - 9G$$

Résumé—La Stabilité des vibrations d'une particule mobile dans un plan et retenue par deux ressorts differents soumis a une tension initiale est etudiée en utilisant la theorie de Floguet et les perturbations. On montre qu'il existe des regions d'instabilite dues aux propriétés differentes des ressorts.

Zusammenfassung—Die Stabilität der Schwingungen eines Teilchens, zwangsbeschränkt auf eine Ebene und von zwei ungleichen Federn mit anfänglicher Spannung gehalten, wird unter Verwendung der Floquet-Theorie mit Störungen untersucht. Es wird gezeigt, dass unerwartete instabile Gebiete als Folge der ungleichen Eigenschaften der Federn auftreten.