ZAMM · Z. angew. Math. Mech. 69 (1989) 9, 275–284

COPPOLA, V. T.; RAND, R. H.

Computer Algebra Implementation of Lie Transforms for Hamiltonian Systems: Application to the Nonlinear Stability of L_4

Wir wenden die Methode der Lie-Transformationen, eine Perturbationsmethode für Differentialgleichungen, auf eine allgemeine Klasse von Hamilton-Systemen an, wobei wir uns der Computer-Algebra bedienen. Dabei entwickeln wir (geschrieben in der Sprache der Koeffizienten im Ausgangssystem H) explizite Formeln zur Transformation des Systems in Birkhoffsche Normalform. Unter Anwendung eines Satzes von Arnold entwickeln wir explizite nichtlineare Stabilitätskriterien allein in der Sprache von H für Systeme, bei denen die lineare Stabilitätsanalyse nicht schlüssig ist. Nach der Bereitstellung zweier Beispiele wenden wir unsere Ergebnisse auf die nichtlineare Stabilität von L_4 an, bei dem es sich um einen Dreieckspunkt im Dreikörperproblem mit zirkularen Restriktionen handelt. In L_4 muß man sich des Satzes von Arnold bedienen, weil es nicht möglich ist, eine Liapunov-Funktion zu finden.

We apply the method of Lie Transforms, a perturbation method for differential equations, to a general class of Hamiltonian systems using computer algebra. In doing so, we develop explicit formulas (written in terms of coefficients in the original system H) for transforming the system into Birkhoff normal form. Using a theorem of Arnold, we develop explicit nonlinear stability criteria solely in terms of H for systems where the linear stability analysis is inconclusive. After providing two examples, we apply our results to the nonlinear stability of L_4, a triangular point in the circular restricted three body problem. At L_4, Arnold's theorem must be used since a Lyapunov function cannot be found.

При помощи компьютерной алгебры применяем метод лиева преобразований, являющий методом возмущения для дифференциальных уравнений, на общий класс систем Гамильтона. При этом развиваем формулы в явном виде (писанны на языке коэффициентов в исходной системе H) для преобразования системы в нормальную форму Биркгофа. С помощью одной теоремы Арнольда развиваем явные нелинейные критерии устойчивости только на языке от H для систем, для которых линейный анализ устойчивости является неубедительным. Обеспечивая два примера применяем наши результаты на нелинейную устойчивость от L_4, являющий треугольной точкой в кругово-ограниченной задаче трех тел. В L_4 надо применять теорему Арнольда, потому-что нельзя найдти функцию Ляпунова.

Introduction

This work concerns Lie transforms, a method for obtaining approximate solutions to systems of differential equations. We apply the method to a general class of two degree of freedom Hamiltonian systems, viz., two coupled nonlinear oscillators with nonresonant frequencies. For systems in this class, we use Lie transforms to approximately reduce the system to an equivalent simpler system which is immediately solvable, i.e., a system with ignorable coordinates.

As an application of our results, we determine the nonlinear stability of the triangular points in the circular restricted three body problem. In doing so we corroborate a computation recently performed by MEYER and SCHMIDT [16]. Their computation was based on their own computer algebra program written in PL/I, whereas the present work is based on readily available utilities written in MACSYMA [19]. Moreover, while their computation was specifically performed for the problem at the triangular point L_4, the present work applies to a problem with arbitrary (symbolic) coefficients.

We begin by introducing the reader to Lie transforms. Then we show how the method may be applied to a particular class of problems, and finally we specialize the results to some examples, including the problem at L_4.

Lie transforms

In this section we summarize the method of Lie transforms (see [8], [12], [15], [17]). This work is concerned with Hamiltonian systems, i.e. systems which are derivable from a single scalar function H, the Hamiltonian:

$$\mathrm{d}x_m/\mathrm{d}t = \partial H/\partial y_m\,, \qquad \mathrm{d}y_m/\mathrm{d}t = -\,\partial H/\partial x_m\,, \tag{1}$$

where x_m and y_m are the dependent variables of the problem, $m = 1, \ldots, N$, where N is called the *number of degrees of freedom*. The method of Lie transforms generates a near-identity transformation from (x_m, y_m) to (X_m, Y_m) variables,

$$\left.\begin{aligned} x_m &= X_m + \text{quadratic terms in } (X_k, Y_k) + \text{cubic terms in } (X_k, Y_k) + \ldots\,, \\ y_m &= Y_m + \text{quadratic terms in } (X_k, Y_k) + \text{cubic terms in } (X_k, Y_k) + \ldots\,, \end{aligned}\right\} \tag{2}$$

which is canonical, i.e., which preserves the Hamiltonian form of the equations:

$$\mathrm{d}X_m/\mathrm{d}t = \partial K/\partial Y_m\,, \quad \mathrm{d}Y_m/\mathrm{d}t = -\,\partial K/\partial X_m\,, \tag{3}$$

where $K = K(X_m, Y_m) = H(x_m, y_m)$ is the Hamiltonian in the new variables (called the *Kamiltonian* after GOLDSTEIN [11]).

The near-identity transformation is generated by first introducing a scaling parameter ε into the problem. Expanding H in a power series about the origin (assumed to be an equilibrium position) yields

$$H = H_0(x_m, y_m) + \varepsilon H_1(x_m, y_m) + \varepsilon^2 H_2(x_m, y_m) + \ldots\,, \tag{4}$$

where $H_n(x_m, y_m)$ is a polynomial of degree $n + 2$. Then the near-identity transformation is generated by the associated Hamiltonian system

$$dx_m/d\varepsilon = \partial W/\partial y_m\,, \qquad dy_m/d\varepsilon = -\,\partial W/\partial x_m\,, \tag{5}$$

in which ε plays the role of time. The transformation evolves in ε, starting with the initial conditions

$$\varepsilon = 0\,, \qquad x_m = X_m\,, \qquad y_m = Y_m\,. \tag{6}$$

The Hamiltonian W of equations (5), called the *generating function,* is also expanded in a power series in ε:

$$W = W_1 + \varepsilon W_2 + \varepsilon^2 W_3 + \dots\,, \tag{7}$$

where W_n is a polynomial of degree $n + 2$. The point of this generating scheme is that the resulting transformation is canonical for any choice of the W_n's (see [8], [15]). The actual choice of these functions depends upon the problem at hand, but the main idea is to pick them so that the new Hamiltonian K is as simple as possible. We note that the parameter ε in this paper corresponds to $-\varepsilon$ in [15] and [19].

The transformation is generated by expanding the variables (x_m, y_m) in Taylor series in ε and using the generating equations (5)—(7) to evaluate the coefficients,

$$x_m = x_m|_{\varepsilon=0} + \left.\frac{dx_m}{d\varepsilon}\right|_{\varepsilon=0} \varepsilon + \left.\frac{d^2x_m}{d\varepsilon^2}\right|_{\varepsilon=0} \frac{\varepsilon^2}{2} + \dots\,, \tag{8}$$

$$x_m|_{\varepsilon=0} = X_m\,, \qquad \left.\frac{dx_m}{d\varepsilon}\right|_{\varepsilon=0} = \left.\frac{\partial W}{\partial y_m}\right|_{\varepsilon=0} = \frac{\partial W_1}{\partial Y_m}, \tag{9), (10}$$

$$\begin{aligned}\left.\frac{d^2x_m}{d\varepsilon^2}\right|_{\varepsilon=0} &= \left.\frac{d}{d\varepsilon}\frac{\partial W}{\partial y_m}\right|_{\varepsilon=0} = \left.\frac{\partial}{\partial\varepsilon}\frac{\partial W}{\partial y_m} + \sum_j \frac{\partial^2 W}{\partial x_j\,\partial y_m}\frac{dx_j}{d\varepsilon} + \frac{\partial^2 W}{\partial y_j\,\partial y_m}\frac{dy_j}{d\varepsilon}\right|_{\varepsilon=0} = \\ &= \frac{\partial W_2}{\partial Y_m} + \sum_j \left.\frac{\partial^2 W}{\partial x_j\,\partial y_m}\frac{\partial W}{\partial y_j} - \frac{\partial^2 W}{\partial y_j\,\partial y_m}\frac{\partial W}{\partial x_j}\right|_{\varepsilon=0} = \frac{\partial W_2}{\partial Y_m} + \sum_j \frac{\partial^2 W_1}{\partial X_j\,\partial Y_m}\frac{\partial W_1}{\partial Y_j} - \frac{\partial^2 W_1}{\partial Y_j\,\partial Y_m}\frac{\partial W_1}{\partial X_j} = \\ &= \frac{\partial W_2}{\partial Y_m} + \left\{\frac{\partial W_1}{\partial Y_m}, W_1\right\},\end{aligned} \tag{11}$$

where the Poisson or Lie bracket $\{f, g\}$ is given by

$$\{f, g\} = \sum_j \frac{\partial f}{\partial X_j}\frac{\partial g}{\partial Y_j} - \frac{\partial f}{\partial Y_j}\frac{\partial g}{\partial X_j}. \tag{12}$$

The transformation is thus found to be given by

$$x_m = X_m + \frac{\partial W_1}{\partial Y_m}\varepsilon + \left[\frac{\partial W_2}{\partial Y_m} + \left\{\frac{\partial W_1}{\partial Y_m}, W_1\right\}\right]\frac{\varepsilon^2}{2} + \dots \tag{13}$$

and similarly,

$$y_m = Y_m - \frac{\partial W_1}{\partial X_m}\varepsilon - \left[\frac{\partial W_2}{\partial X_m} + \left\{\frac{\partial W_1}{\partial X_m}, W_1\right\}\right]\frac{\varepsilon^2}{2} + \dots\,. \tag{14}$$

In order to obtain the transformed Kamiltonian K (cf. (3)), the transformation (13), (14) is substituted into a power series expansion for the original Hamiltonian H:

$$K(X_m, Y_m) = H(x_m, y_m) = H_0(x_m, y_m) + \varepsilon H_1(x_m, y_m) + \varepsilon^2 H_2(x_m, y_m) + \dots\,, \tag{15}$$

$$H_0(x_m, y_m) = H_0\left(X_m + \frac{\partial W_1}{\partial Y_m}\varepsilon + \dots,\ Y_m - \frac{\partial W_1}{\partial X_m}\varepsilon - \dots\right) = H_0|_{\varepsilon=0} + \left.\frac{dH_0}{d\varepsilon}\right|_{\varepsilon=0}\varepsilon + \left.\frac{d^2H_0}{d\varepsilon^2}\right|_{\varepsilon=0}\frac{\varepsilon^2}{2} + \dots\,, \tag{16}$$

$$H_0|_{\varepsilon=0} = H_0(X_m, Y_m)\,, \tag{17}$$

$$\left.\frac{dH_0}{d\varepsilon}\right|_{\varepsilon=0} = \sum_j \left.\frac{\partial H_0}{\partial x_j}\frac{dx_j}{d\varepsilon} + \frac{\partial H_0}{\partial y_j}\frac{dy_j}{d\varepsilon}\right|_{\varepsilon=0} = \sum_j \left.\frac{\partial H_0}{\partial x_j}\frac{\partial W}{\partial y_j} - \frac{\partial H_0}{\partial y_j}\frac{\partial W}{\partial x_j}\right|_{\varepsilon=0} = \sum_j \frac{\partial H_0}{\partial X_j}\frac{\partial W_1}{\partial Y_j} - \frac{\partial H_0}{\partial Y_j}\frac{\partial W_1}{\partial X_j} = \{H_0, W_1\}\,, \tag{18}$$

where the generating equations (5)—(7) have been used. This gives

$$H_0(x_m, y_m) = H_0(X_m, Y_m) + [\{H_0, W_1\}\,\varepsilon + \{\{H_0, W_1\}, W_1\} + \{H_0, W_2\}]\frac{\varepsilon^2}{2} + \dots\,. \tag{19}$$

This equation, which represents the expansion of H_0 under the near-identity transformation (13), (14), also holds for any of the H_n's, and in fact is valid for any function $f(x_m, y_m)$. Substitution of (19) and the corresponding equations on the other $H_n(x_m, y_m)$ into (15) gives, after some simplification:

$$K(X_m, Y_m) = K_0(X_m, Y_m) + K_1(X_m, Y_m)\,\varepsilon + K_2(X_m, Y_m)\,\varepsilon^2 + \dots\,, \tag{20}$$

where

$$K_0 = H_0 , \qquad K_1 = H_1 + \{H_0, W_1\} , \qquad K_2 = H_2 + \tfrac{1}{2}\{H_0, W_2\} + \tfrac{1}{2}\{K_1, W_1\} + \tfrac{1}{2}\{H_1, W_1\} , \quad (21), (22), (23)$$

$$K_3 = H_3 + \tfrac{1}{3}\{H_0, W_3\} + \tfrac{1}{3}\{K_1, W_2\} + \tfrac{1}{3}\{K_2, W_1\} + \tfrac{1}{6}\{H_1, W_2\} + \tfrac{2}{3}\{H_2, W_1\} + \tfrac{1}{6}\{\{H_1, W_1\}, W_1\} , \quad (24)$$

$$K_4 = H_4 + \tfrac{1}{4}\{H_0, W_4\} + \tfrac{1}{4}\{K_1, W_3\} + \tfrac{1}{4}\{K_2, W_2\} + \tfrac{1}{4}\{K_3, W_1\} + \tfrac{1}{12}\{H_1, W_3\} + \tfrac{1}{4}\{H_2, W_2\} + \tfrac{3}{4}\{H_3, W_1\} + \tfrac{1}{12}\{\{H_1, W_1\}, W_2\} + \tfrac{1}{24}\{\{H_1, W_2\}, W_1\} + \tfrac{1}{4}\{\{H_2, W_1\}, W_1\} + \tfrac{1}{24}\{\{\{H_1, W_1\}, W_1\}, W_1\} . \quad (25)$$

In equations (21)—(25), the H_n and W_n are taken as functions of the variables X_m, Y_m.

So we see that the method of Lie transforms is nothing more than the introduction of the generating equations (5)—(7) into Taylor series expansions for the variables (x_m, y_m) and H. However, the transformation equations (e.g. (21)—(25)) can be generated much more efficiently than by the foregoing expansion method. There are several schemes for doing so (including the original method of DEPRIT [8] based on the "Lie triangle" and a method of DRAGT and FINN [10] based on infinite products rather than infinite series), but we prefer the following method (see [15]), which is easily implemented on MACSYMA ([13], [14], [19]).

Define the operators L_n and S_n as follows:

$$L_n = \{\ , W_n\} , \quad (26)$$

$$S_0 = Id \text{ (the identity operator)}, \quad S_n = \frac{1}{n}\sum_{m=0}^{n-1} L_{n-m}S_m , \qquad n = 1, 2, 3, \dots . \quad (27.1), (27.2)$$

Then the near-identity transformation from (x_m, y_m) to (X_m, Y_m) variables is given by

$$x_m = [S_0 + \varepsilon S_1 + \varepsilon^2 S_2 + \dots]\, X_m , \qquad y_m = [S_0 + \varepsilon S_1 + \varepsilon^2 S_2 + \dots]\, Y_m \quad (28.1), (28.2)$$

and the n-th term K_n of the Kamiltonian is given by the expression

$$K_n = H_n + \frac{1}{n}\{H_0, W_n\} + \frac{1}{n}\sum_{m=1}^{n-1} [L_{n-m}K_m + mS_{n-m}H_m] , \qquad n = 2, 3, 4, \dots , \quad (29)$$

where the cases $n = 0, 1$ are given by equations (21), (22).

Coupled oscillators

In this work we shall apply the method of Lie transforms to two degree of freedom Hamiltonian systems in which H_0 has the special form:

$$H_0 = \tfrac{1}{2}(p_1^2 + \omega_1^2 q_1^2) - \tfrac{1}{2}(p_2^2 + \omega_2^2 q_2^2) , \quad (30)$$

where q_m and p_m are variables representing the displacement and momentum of oscillator m. For $\varepsilon = 0$, the equations of motion corresponding to such a Hamiltonian become

$$\dot q_m = p_m \quad\text{and}\quad \dot p_m = -\omega_m^2 q_m , \quad\text{or}\quad \ddot q_m + \omega_m^2 q_m = 0 . \quad (31)$$

Thus when $\varepsilon = 0$, the system has eigenvalues $\pm i\omega_1$, $\pm i\omega_2$, where $i = \sqrt{-1}$, and we change variables to eigencoordinates (x_m, y_m),

$$q_m = \frac{x_m}{\omega_m} + i\frac{y_m}{2} , \qquad p_m = \frac{\omega_m y_m}{2} + i x_m \quad (32)$$

for which the equations of motion (31) and Hamiltonian (30) take the form

$$\dot x_m = i\omega_m x_m \quad\text{and}\quad \dot y_m = -i\omega_m y_m , \quad (33)$$

$$H_0 = i\omega_1 x_1 y_1 - i\omega_2 x_2 y_2 . \quad (34)$$

In these coordinates, each H_n becomes a polynomial of degree $n + 2$ in the four variables x_1, y_1, x_2, y_2. For example, there are 20 cubic monomials which form a basis for H_1:

$$H_1 = \text{linear combination of } \{x_1^3, x_1^2x_2, x_1^2y_1, x_1^2y_2, x_1x_2^2, x_1x_2y_1, x_1x_2y_2, x_1y_1^2, x_1y_1y_2, x_1y_2^2, x_2^3, x_2^2y_1, x_2^2y_2, x_2y_1^2, x_2y_1y_2, x_2y_2^2, y_1^3, y_1^2y_2, y_1y_2^2, y_2^3\} . \quad (35)$$

The numbers of basis monomials for H_2, H_3 and H_4 are:

Term	Degree	No. of basis monomials
H_1	3	20
H_2	4	35
H_3	5	56
H_4	6	84

We now come to the question of how to choose the generating functions W_n so as to best simplifiy the Kamiltonians K_n. At the n-th step of the method, K_n is given by equation (29),

$$K_n = \frac{1}{n}\{H_0, W_n\} + \text{terms which are already known}\,. \tag{36}$$

Now with H_0 in the simplified form (34),

$$\{H_0, W_n\} = \frac{\partial H_0}{\partial X_1}\frac{\partial W_n}{\partial Y_1} - \frac{\partial H_0}{\partial Y_1}\frac{\partial W_n}{\partial X_1} + \frac{\partial H_0}{\partial X_2}\frac{\partial W_n}{\partial Y_2} - \frac{\partial H_0}{\partial Y_2}\frac{\partial W_n}{\partial X_2} = i\omega_1\left[Y_1\frac{\partial W_n}{\partial Y_1} - X_1\frac{\partial W_n}{\partial X_1}\right] - i\omega_2\left[Y_2\frac{\partial W_n}{\partial Y_2} - X_2\frac{\partial W_n}{\partial X_2}\right] \tag{37}$$

we want to choose W_n so that this linear partial differential operator on W_n cancels as many terms as possible in (36). Each term to be cancelled will be of the form

$$A X_1^j Y_1^l X_2^r Y_2^s \tag{38}$$

where A is a constant. In view of the linearity of (37), we choose W_n to be a sum of terms, one for each term (38) to be cancelled, of the form

$$W_n = B X_1^j Y_1^l X_2^r Y_2^s \tag{39}$$

where B is an undetermined constant. Then

$$\frac{1}{n}\{H_0, W_n\} = \frac{1}{n}(i\omega_1(l-j) - i\omega_2(s-r))\, B X_1^j Y_1^l X_2^r Y_2^s \tag{40}$$

leading to the choice

$$B = \frac{iAn}{\omega_1(l-j) - \omega_2(s-r)}\,, \qquad n = j + l + r + s - 2\,. \tag{41}$$

Note that this scheme fails if the denominator of (41) vanishes. Assuming that the frequencies ω_1 and ω_2 are incommensurable (nonresonant), the denominator will vanish only if both

$$l = j \quad \text{and} \quad s = r\,. \tag{42}$$

Thus we cannot remove terms of the form

$$(X_1Y_1)^j\,(X_2Y_2)^r\,. \tag{43}$$

This means that we can always reduce every such (nonresonant) problem to the form:

$$K_0 = H_0 = i\omega_1(X_1Y_1) - i\omega_2(X_2Y_2)\,, \qquad K_1 = 0\,, \tag{44), (45}$$

$$K_2 = K_{2200}(X_1Y_1)^2 + K_{1111}(X_1Y_1)\,(X_2Y_2) + K_{0022}(X_2Y_2)^2\,, \qquad K_3 = 0\,, \tag{46), (47}$$

$$K_4 = K_{3300}(X_1Y_1)^3 + K_{2211}(X_1Y_1)^2\,(X_2Y_2) + K_{1122}(X_1Y_1)\,(X_2Y_2)^2 + K_{0033}(X_2Y_2)^3\,. \tag{48}$$

That is, every such nonresonant two degree of freedom problem can, to $O(4)$, be reduced to only 7 coefficients. Note that in this case the resulting Kamiltonian is a function only of the "action" variables,

$$I_1 = iX_1Y_1 \quad \text{and} \quad I_2 = iX_2Y_2 \tag{49}$$

and hence both coordinates are ignorable and the system is immediately solveable to $O(4)$. Such a system is said *to be in Birkhoff normal form* ([5], p. 85).

By inspection of (41), the foregoing scheme fails at special resonant values of ω_1 and ω_2. In solving for W_n, resonant terms occur for integer values of k_1 and k_2 such that

$$k_1\omega_1 + k_2\omega_2 = 0\,, \qquad |k_1| + |k_2| \leqq n + 2\,. \tag{50}$$

In such cases additional non-removable terms occur. We shall not consider such resonant cases in this work.

Computer algebra

The computation just described turns out to involve vast quantities of algebra. We used the computer algebra system MACSYMA ([18]) in order to do the computation more accurately and more efficiently than by hand. For example, the key formulas (12), (26), (27), (29) can be represented in MACSYMA via the following lines of code ([7], [19]):

```
POISSON(F,G):=
SUM(DIFF(F,X[I])*DIFF(G,Y[I])-DIFF(F,Y[I])*DIFF(G,X[I]),I,1,N)$
L(I,F):=POISSON(F,W[I])$
S(I,F):=(IF I=0 THEN F ELSE SUM(L(I-M,S(M,F)),M,O,I-1)/I)$
K[I]:=(H[I]+POISSON(H[O],W[I])/I
        +SUM(L(I-M,K[M])+M*S(I-M,H[M]),M,1,I-1)/I)$
```

In order to efficiently compute W_n by the formulas (39), (41), we use the MACSYMA tool called *pattern matching*. A rule named WSOLVE is defined as follows:

```
LET(X1^J*Y1^L*X2^R*Y2^S,
    X1^J*Y1^L*X2^R*Y2^S*%I*N/(W1*(L-J)-W2*(S-R)),WSOLVE)$
```

That is, replace the term $X_1^jY_1^lX_2^rY_2^s$ by $\dfrac{inX_1^jY_1^lX_2^rY_2^s}{\omega_1(l-j)-\omega_2(s-r)}$. When WSOLVE is applied to the "terms which are already known" on the right hand side of equation (36), the correct expression for W_n is automatically generated. Note that this rule is not applied to non-removable terms of the form (43).

One could hope to simply apply these formulas to the problem at hand, and to thereby automatically obtain the transformed Kamiltonian. Unfortunately, the size of the $O(4)$ computation is too large to proceed directly; MACSYMA on a Symbolics 3670 runs out of space. E.g. from (25) we see that the computation of K_4 involves the evaluation of the quantity $\{\{\{H_1, W_1\}, W_1\}, W_1\}$. The innermost Poisson bracket involves 20 terms for H_1 and 20 terms for W_1, i.e. 400 pairs which can be collected together into 35 terms (since there are 35 fourth degree basis monomials). These then need to be combined with the 20 terms of W_1 in order to evaluate the second Poisson bracket, i.e. 700 pairs which combine together into 56 terms. Next the third Poisson bracket combines the previous result with the 20 terms of W_1 to require the computation of 1120 pairs, which may be collected together into 84 terms.

In order to complete the computation, we broke it up into pieces, each of which was sufficiently small so as not to cause MACSYMA to encounter space problems. We shall refer to our strategy for treating such large computations as the *method of telescoping compositions*. As an example of this strategy, we once again consider the computation of the triple Poisson bracket $\{\{\{H_1, W_1\}, W_1\}, W_1\}$. We first compute $\{H_1, W_1\}$ and store the resulting 35 coefficients A_{jlrs} in a disk file. Next, instead of computing $\{\{H_1, W_1\}, W_1\}$, we compute instead $\{A, W_1\}$, where A is a dummy polynomial with symbolic coefficients A_{jlrs}. Although' we are eventually interested in identifying these coefficients with those we have stored in a disk file, we save that step for later. We store the resulting 56 coefficients B_{jlrs} of $\{A, W_1\}$ in a disk file. Next we compute $\{B, W_1\}$, where now B is a dummy polynomial with symbolic coefficients B_{jlrs}. This results in 84 coefficients which are known in terms of the B_{jlrs} coefficients. The latter are stored in a file and are known in terms of the A_{jlrs} coefficients, which are also stored in a disk file. At this point the computation of $\{\{\{H_1, W_1\}, W_1\}, W_1\}$ is complete, although it still remains to plug the values of the A_{jlrs} and B_{jlrs} coefficients into the final result. — For a complete listing of the programs, see [7].

Results

The results of this work take the form of expressions for the transformed Kamiltonian K in terms of the original Hamiltonian H. If we express H in x_m, y_m eigencoordinates defined by equations (32), then H_0 takes the canonical form (34), and the polynomials H_n of (4) can be written as

$$H_n = \sum' H_{jlrs}x_1^jy_1^lx_2^ry_2^s\,, \qquad n = j + l + r + s - 2\,,$$

where the H_{jlrs} are given constants. Then the coefficients K_{jlrs} in K_2 in equation (46) are given by:

$$K_{2200} = H_{2200} + i\left(\frac{-1}{\omega_2}H_{1101}H_{1110} + \frac{3}{\omega_1}\{H_{1200}H_{2100} + H_{0300}H_{3000}\} - \right.$$
$$\left. - \frac{1}{2\omega_1+\omega_2}H_{0210}H_{2001} + \frac{1}{2\omega_1-\omega_2}H_{0201}H_{2010}\right), \tag{51}$$

$$K_{1111} = H_{1111} + i\left(\frac{2}{\omega_1}\{H_{1011}H_{1200} + H_{0111}H_{2100}\} - \frac{4}{\omega_1+2\omega_2}H_{0120}H_{1002} - \right.$$
$$- \frac{2}{\omega_2}\{H_{1101}H_{0021} + H_{1110}H_{0012}\} + \frac{4}{\omega_2+2\omega_1}H_{0210}H_{2001} -$$
$$\left. - \frac{4}{2\omega_2-\omega_1}H_{0102}H_{1020} + \frac{4}{2\omega_1-\omega_2}H_{0201}H_{2010}\right), \tag{52}$$

$$K_{0022} = H_{0022} + i\left(\frac{1}{\omega_1}H_{0111}H_{1011} - \frac{3}{\omega_2}\{H_{0003}H_{0030} + H_{0012}H_{0021}\} + \right.$$
$$\left. + \frac{1}{2\omega_2+\omega_1}H_{0120}H_{1002} - \frac{1}{2\omega_2-\omega_1}H_{0102}H_{1020}\right). \tag{53}$$

The comparable coefficients in K_4 in (48) were also found, but cannot be displayed here because they are too long. E.g., the ASCII files for K_{3300} and K_{0033} contain 164K characters, while those for K_{1122} and K_{2211} contain 468K. These expressions simplify greatly, however, in the special case in which H_1 and H_3 are identically zero. Since this special case occurs in frequently in sample problems, we give the associated coefficients of K_4 here:

$$K_{3300} = H_{3300} - \frac{iH_{0301}H_{3010}}{\omega_2 - 3\omega_1} - \frac{iH_{1201}H_{2110}}{\omega_2 - \omega_1} - \frac{iH_{1210}H_{2101}}{\omega_2 + \omega_1} - \frac{iH_{0310}H_{3001}}{\omega_2 + 3\omega_1} + \frac{4iH_{0100}H_{4000}}{\omega_1} + \frac{4iH_{1300}H_{3100}}{\omega_1}, \tag{54}$$

$$\begin{aligned} K_{2211} = H_{2211} &- \frac{9iH_{0301}H_{3010}}{\omega_2 - 3\omega_1} - \frac{i(3H_{1201} + 2H_{0112})\, H_{2110}}{\omega_2 - \omega_1} + \frac{9iH_{0310}H_{3001}}{\omega_2 + 3\omega_1} + \frac{i(3H_{1210} - 2H_{0121})\, H_{2101}}{\omega_2 + \omega_1} - \\ &- \frac{2iH_{0202}H_{2020}}{\omega_2 - \omega_1} - \frac{2iH_{1021}H_{1201}}{\omega_2 - \omega_1} - \frac{2iH_{0220}H_{2002}}{\omega_2 + \omega_1} - \frac{2iH_{1012}H_{1210}}{\omega_2 + \omega_1} - \frac{2iH_{1102}H_{1120}}{\omega_2} + \\ &+ \frac{3iH_{0211}H_{3100}}{\omega_1} + \frac{3iH_{1300}H_{2011}}{\omega_1}, \end{aligned} \tag{55}$$

$$\begin{aligned} K_{1122} = H_{1122} &- \frac{9iH_{0103}H_{1030}}{3\omega_2 - \omega_1} - \frac{i(3H_{0112} + 2H_{1201})\, H_{1021}}{\omega_2 - \omega_1} - \frac{9iH_{0130}H_{1003}}{3\omega_2 + \omega_1} - \frac{i(3H_{0121} - 2H_{1210})H_{1012}}{\omega_2 + \omega_1} - \\ &- \frac{2iH_{0202}H_{2020}}{\omega_2 - \omega_1} - \frac{2iH_{0112}H_{2110}}{\omega_2 - \omega_1} + \frac{2iH_{0220}H_{2002}}{\omega_2 + \omega_1} + \frac{2iH_{0121}H_{2101}}{\omega_2 + \omega_1} + \frac{2iH_{0211}H_{2011}}{\omega_1} - \\ &- \frac{3iH_{0013}H_{1120}}{\omega_2} - \frac{3iH_{0031}H_{1102}}{\omega_2}, \end{aligned} \tag{56}$$

$$K_{0033} = H_{0033} - \frac{iH_{0103}H_{1030}}{3\omega_2 - \omega_1} + \frac{iH_{0130}H_{1003}}{3\omega_2 + \omega_1} - \frac{iH_{0112}H_{1021}}{\omega_2 - \omega_1} + \frac{iH_{0121}H_{1012}}{\omega_2 + \omega_1} - \frac{4iH_{0004}H_{0040}}{\omega_2} - \frac{4iH_{0013}H_{0031}}{\omega_2}. \tag{57}$$

Arnold's theorem

We are interested in applying the previous results to the determination of the stability of the equilibrium at the origin in a system of two nonlinear coupled oscillators in which H_0 has the form (30). Note that the linearized Hamiltonian differential equations (1) corresponding to $H = H_0$ have purely imaginary eigenvalues, and thus are inconclusive regarding stability. Moreover, because of the minus sign in (30), H_0 is not positive definite, and Lyapunov's direct method [7] cannot be used to determine stability.

For such cases, stability may be determined by appealing to a theorem of Arnold [4], which has been restated and reproved by Meyer and Schmidt [16]. The theorem, based on the existence of invariant tori in KAM theory [3], gives sufficient conditions for stability in nonresonant systems, in terms of the transformed Hamiltonian $K(I_1, I_2)$ which has been put in Birkhoff normal form, cf. (49). The terms K_n of equations (44)—(48) are thought of as functions of I_1 and I_2, $K_n(I_1, I_2)$. The theorem involves quantities D_n defined by

$$D_n = K_n(\omega_2, \omega_1)\,. \tag{58}$$

From (44)—(48), the first two non-identically zero D_n's are D_2 and D_4:

$$D_2 = -(K_{2200}\omega_2^2 + K_{1111}\omega_1\omega_2 + K_{0022}\omega_1^2)\,, \qquad D_4 = i(IK_{3300}\omega_2^3 + K_{2211}\omega_1\omega_2^2 + K_{1122}\omega_1^2\omega_2 + K_{0033}\omega_1^3)\,. \tag{59), (60}$$

Arnold's theorem states that the origin is stable for those parameter values for which $D_2 \neq 0$. In the case that $D_2 = 0$, stability is assured if $D_4 \neq 0$, and so on. I.e., the origin is stable if $D_{2n} \neq 0$ for some n.

Using the expressions (51)—(57) for the coefficients K_{jlrs}, expressions for D_2 and D_4 (the latter in the special case that $H_1 = H_3 = 0$) may be obtained:

$$\begin{aligned} D_2 = &-(\omega_2^2 H_{2200} + \omega_1\omega_2 H_{1111} + \omega_1^2 H_{0022}) + \\ &+ i\Big[\omega_2 H_{1101}H_{1110} - \omega_1 H_{1011}H_{0111} + 2\omega_1(H_{1110}H_{0012} + H_{1101}H_{0021}) - 2\omega_2(H_{0111}H_{2100} + H_{1011}H_{1200}) + \\ &+ \frac{3\omega_1^2}{\omega_2}(H_{0003}H_{0030} + H_{0021}H_{0012}) - \frac{3\omega_2^2}{\omega_1}(H_{3000}H_{0300} + H_{2100}H_{1200}) + \\ &+ \frac{4\omega_2 + \omega_1}{2\omega_2 - \omega_1}\omega_1 H_{1020}H_{0102} + \frac{4\omega_2 - \omega_1}{2\omega_2 + \omega_1}\omega_1 H_{1002}H_{0120} - \\ &- \frac{4\omega_1 + \omega_2}{2\omega_1 - \omega_2}\omega_2 H_{2010}H_{0201} - \frac{4\omega_1 - \omega_2}{2\omega_1 + \omega_2}\omega_2\, H_{2001}H_{0210}\Big], \end{aligned} \tag{61}$$

$$\begin{aligned}D_4 = {} & i(\omega_2^3 H_{3300} + \omega_2^2\omega_1 H_{2211} + \omega_2\omega_1^2 H_{1122} + \omega_1^3 H_{0033}) - \\ & - 2\omega_1\omega_2(H_{1102}H_{1120} + H_{0211}H_{2011} + H_{0202}H_{2020} + H_{1021}H_{1201} + H_{1012}H_{1210} + H_{0220}H_{2002} + \\ & + H_{0112}H_{2110} + H_{2101}H_{0121}) - 3\omega_2^2(H_{3100}H_{0211} + H_{1300}H_{2011}) - 3\omega_1^2(H_{0013}H_{1120} + H_{0031}H_{1102}) - \\ & - 4\frac{\omega_1^3}{\omega_2}(H_{0004}H_{0040} + H_{0013}H_{0031}) - 4\frac{\omega_2^3}{\omega_1}(H_{4000}H_{0400} + H_{1300}H_{3100}) - \\ & - \frac{\omega_2^2 H_{1201}H_{2110}(\omega_2 + 3\omega_1)}{\omega_2 + \omega_1} - \frac{\omega_2^2 H_{1210}H_{2101}(\omega_2 - 3\omega_1)}{\omega_2 - \omega_1} - \\ & - \frac{\omega_1^2 H_{1021}H_{0112}(\omega_1 + 3\omega_2)}{\omega_1 + \omega_2} - \frac{\omega_1^2 H_{1012}H_{0121}(\omega_1 - 3\omega_2)}{\omega_1 - \omega_2} - \\ & - \frac{\omega_2^2 H_{0301}H_{3010}(\omega_2 + 9\omega_1)}{\omega_2 + 3\omega_1} - \frac{\omega_2^2 H_{0310}H_{3001}(\omega_2 - 9\omega_1)}{\omega_2 - 3\omega_1} - \\ & - \frac{\omega_1^2 H_{0103}H_{1030}(\omega_1 + 9\omega_2)}{\omega_1 + 3\omega_2} - \frac{\omega_1^2 H_{0130}H_{1003}(\omega_1 - 9\omega_2)}{\omega_1 - 3\omega_2}\end{aligned} \quad (62)$$

(assumes $H_1 = H_3 = 0$).

The expression for D_4 in the general case is too long to be included here, but is available on our computer for numerical evaluation.

Example 1

We consider a variation of the Henon-Heiles Hamiltonian where the linear oscillators are not at low-order resonance and are of different signs:

$$H = \tfrac{1}{2}(p_1^2 + \omega^2 q_1^2) - \tfrac{1}{2}(p_2^2 + q_2^2) + q_1^2 q_2 - \tfrac{1}{3} q_2^3 . \quad (63)$$

Using the transformation to eigencoordinates given by equation (32), H becomes

$$\begin{aligned}H = {} & i\omega x_1 y_1 - i x_2 y_2 - \frac{1}{3} x_2^3 + \frac{i}{24} y_2^3 + \frac{i}{2\omega^2} x_1^2 y_2 + \frac{1}{\omega^2} x_1^2 x_2 - \frac{i}{2} x_2^2 y_2 - \frac{1}{4} y_1^2 x_2 - \frac{i}{8} y_1^2 y_2 + \frac{1}{4} x_2 y_2^2 + \\ & + \frac{i}{\omega} x_1 x_2 y_1 - \frac{1}{2\omega} x_1 y_1 y_2 , \qquad \omega > 0 .\end{aligned} \quad (64)$$

Then using equations (51)—(53), we find the K_2 coefficients to be

$$K_{2200} = \frac{3 - 8\omega^2}{4\omega^2(4\omega^2 - 1)}, \qquad K_{1111} = \frac{4\omega^2 + 1}{\omega(4\omega^2 - 1)}, \qquad K_{0022} = -\frac{5}{12} . \quad (65), (66), (67)$$

Using equation (61), we find D_2 to be

$$D_2 = \frac{20\omega^6 - 53\omega^4 + 12\omega^2 - 9}{12\omega^2(4\omega^2 - 1)} . \quad (68)$$

We find that $D_2 = 0$ only for $\omega = \omega_c \simeq 1.5752078 \ldots$ In order to determine the stability of the origin for $\omega = \omega_c$, we must consider the D_4 condition. Because H contains only cubic nonlinear terms and because each cubic coefficient is simple, we are able to find the expression for D_4 algebraically. The coefficients for K_4 turn out to be

$$K_{3300} = \frac{i(1024\omega^8 + 768\omega^6 - 1632\omega^4 + 596\omega^2 - 51)}{48\omega^5(2\omega - 1)^3 (2\omega + 1)^3}, \qquad K_{2211} = \frac{i(384\omega^{10} - 288\omega^8 + 16\omega^6 - 340\omega^4 + 159\omega^2 - 6)}{4\omega^4(1 - \omega^2)(2\omega - 1)^3 (2\omega + 1)^3}, \quad (69), (70)$$

$$K_{1122} = \frac{i(4\omega^2 + 1)(320\omega^8 - 480\omega^6 + 360\omega^4 - 161\omega^2 + 6)}{12\omega^3(\omega^2 - 1)(2\omega - 1)^3 (2\omega + 1)^3}, \qquad K_{0033} = -\frac{235i}{432} \quad (71), (72)$$

and D_4 becomes

$$D_4 = \frac{(15040\omega^{16} - 72400\omega^{14} + 113172\omega^{12} - 77935\omega^{10} + 14491\omega^8 - 10188\omega^6 - 3096\omega^4 + 5175\omega^2 - 459)}{432\omega^5(\omega^2 - 1)(2\omega - 1)^3 (2\omega + 1)^3} . \quad (73)$$

So, at $\omega = \omega_c$, $D_4 = -0.19180289 \ldots \neq 0$.

Thus by Arnold's theorem, the origin is nonlinearly stable. We note that this result does not apply to a small set of resonant values of ω which correspond to vanishing denominators in the algorithm (41). From equation (50) with $n = 2$, we find the following resonant values of ω:

$$\omega = \{\tfrac{1}{3}, \tfrac{1}{2}, 1, 2, 3\} .$$

Example 2

This second example involves a spinning mass-spring system, which contains no odd powered terms in the Hamiltonian. Consider 4 identical springs, each attached at one end to the outer rim of a wheel of unit radius separated by 90°. The other end of each spring is attached to a unit mass which is free to move about its equilibrium position at the center, see Fig. 1. Let the $Q_1 - Q_2$ axes rotate with the wheel with angular velocity $\omega > 0$ relative to an inertial frame. Each spring is unstretched when the mass is at the origin.

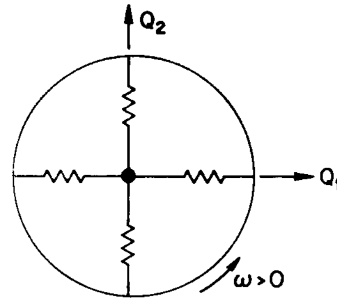

Fig. 1. Example 2 involves a spinning mass-spring system. The unit mass is restrained by 4 identical nonlinear springs. The $Q_1 - Q_2$ axes are fixed to the wheel and rotate relative to an inertial frame

The potential energy V_i for each spring under a deflection δ_i is taken to be

$$V_i = \tfrac{1}{4}\,(\delta_i^2 + \mu\delta_i^4) \tag{74}$$

where the linear spring constant has been taken equal to $\frac{1}{2}$ and μ is a nonlinear spring constant. Then this system has the Hamiltonian

$$H = \tfrac{1}{2}\,(P_1^2 + P_2^2) + \omega(P_1Q_2 - P_2Q_1) + V_1 + V_2 + V_3 + V_4 \tag{75}$$

where P_i are momenta. Then upon taking the Taylor series of H about the origin, H becomes [7]

$$H = \tfrac{1}{2}(P_1^2 + P_2^2) + \omega(P_1Q_2 - P_2Q_1) + \tfrac{1}{2}(Q_1^2 + Q_2^2) + \tfrac{1}{8}\,[(4\mu + 1)\,Q_1^4 - 8Q_1^2Q_2^2 + (4\mu + 1)\,Q_2^4] - \\ - \tfrac{1}{16}\,[Q_1^6 + 4\,(\mu - 1)Q_1^4Q_2^2 + 4(\mu - 1)Q_1^2Q_2^4 + Q_2^6] + \ldots = H_0 + H_2 + H_4 + \ldots \tag{76}$$

Using the linear differential equations corresponding to H_0, we find the characteristic equation to be

$$\lambda^4 + 2\lambda^2(1 + \omega^2) + (\omega^2 - 1)^2 = 0 \tag{77}$$

which has eigenvalues $\lambda = \{\pm\, i(1 - \omega), \pm\, i(1 + \omega)\}$. From this we conclude that the equilibrium at the origin is elliptic for $\omega \neq 1$, i.e., comprised of two oscillators with frequencies $1 - \omega$ and $1 + \omega$ in the first approximation. Then using a canonical eigenvector transformation from (Q_m, P_m) to (x_m, y_m) gives [7]

$$H_0 = i(1 - \omega)\,x_1y_1 - i(1 + \omega)\,x_2y_2 \tag{78}$$

which is in the proper form for our analysis. After similarly transforming H_2 and H_4, we use equations (51)–(57) to find that

$$K_{2200} = K_{0022} = \frac{1 - 12\mu}{32}\,, \qquad K_{1111} = \frac{1 - 12\mu}{8}\,; \tag{79}$$

$$K_{3300} = \frac{i[(576\mu^2 - 32\mu + 20)\,\omega^2 - (864\mu^2 - 48\,\mu + 30)\,\omega + 272\mu^2 - 56\mu - 15]}{1024(\omega - 1)\,(2\omega - 1)}\,, \tag{80}$$

$$K_{2211} = \frac{3i[(1440\mu^2 - 48\mu + 58)\,\omega^2 - (720\mu^2 - 24\mu + 29)\,\omega - 48\mu^2 - 120\mu - 75]}{1024\omega(2\omega - 1)}\,, \tag{81}$$

$$K_{1122} = \frac{3i[(1440\mu^2 - 48\mu + 58)\,\omega^2 + (720\mu^2 - 24\mu + 29)\,\omega - 48\mu^2 - 120\mu - 75]}{1024\omega(2\omega + 1)}\,, \tag{82}$$

$$K_{0033} = \frac{i[(576\mu^2 - 32\mu + 20)\,\omega^2 + (864\mu^2 - 48 + 30)\,\omega + 272\mu^2 - 56\mu - 15]}{1024(\omega + 1)\,(2\omega + 1)}\,. \tag{83}$$

We find D_2 and D_4 using eqs. (61)–(62) to be

$$D_2 = \frac{(12\mu - 1)\,(3\omega^2 - 1)}{16}\,, \tag{84}$$

$$D_4 = \frac{\left[\begin{array}{c}(9792\mu^2 - 352\mu + 388)\,\omega^8 - (17744\mu^2 + 264\mu + 1213)\,\omega^6 + (9104\mu^2 + 232\mu + 657)\,\omega^4 - \\ - (1392\mu^2 + 216\mu + 207)\,\omega^2 - 144\mu^2 - 360\mu - 225]\end{array}\right]}{512\omega(\omega^2 - 1)\,(4\omega^2 - 1)}\,. \tag{85}$$

Then $D_2 = 0$ for $\mu = \frac{1}{12}$ and $\omega^2 = \frac{1}{3}$, which are two lines in the $\mu - \omega$ parameter plane. When $D_2 = 0$ we must check the D_4 condition. Consider the line $\mu = \frac{1}{12}$. The value of D_4 on this line is

$$D_4\left(\mu = \frac{1}{12}\right) = \frac{60\omega^8 - 191\omega^6 + 104\omega^4 - 33\omega^2 - 36}{72\omega(\omega^2 - 1)\,(4\omega^2 - 1)} \tag{86}$$

which is zero only for $\omega = \omega_c \simeq 1.6241875$ ... Now consider the line $\omega^2 = \frac{1}{3}$. D_4 on this line becomes

$$D_4\left(\omega^2 = \frac{1}{3}\right) = \frac{\sqrt{3}}{288}\,(336\mu^2 + 1064\mu + 661) \tag{87}$$

which is zero only for $\mu = \mu_{1,2} = -\frac{1}{84}\,(133 \pm 4\,\sqrt{238}) \simeq \{-0.84870, -2.31796\}$.

We now apply the stability theorem. First, note that we consider $\omega > 0$ and that for $\omega = 1$ the origin is not elliptic so that our analysis does not apply there. From equation (50) with $n = 2$, we must also exclude $\omega = \{\frac{1}{3}, \frac{1}{2}, 2, 3\}$ from the analysis. Applying the D_2 condition, we find that the origin is stable everywhere in the $\mu - \omega$ parameter plane except possibly along the two lines $\mu = \frac{1}{12}$ and $\omega^2 = \frac{1}{3}$. On these lines the D_4 condition must be used. From (50) with $n = 4$, we must now exclude $\omega = \{\frac{1}{5}, \frac{3}{5}, \frac{2}{3}, \frac{3}{2}, \frac{5}{3}, 5\}$ when $\mu = \frac{1}{12}$. Elsewhere on $\mu = \frac{1}{12}$, the origin is stable provided $\omega \neq \omega_c$; on $\omega^2 = \frac{1}{3}$, the origin is stable provided $\mu \neq \mu_{1,2}$. For the three points where $D_2 = D_4 = 0$, the D_6 condition must be used to prove nonlinear stability.

We note that for $\omega < 1$, stability of the origin can be independently proved by Lyapunov's direct method [7].

Application to the problem of three bodies

The circular restricted three body problem is well-known to exhibit five equilibria in a rotating barycentric coordinate system [20]. L_1, L_2 and L_3 represent equilibrium positions of the third body, in which all three bodies are collinear. All three of these are unstable for all values of the mass ratio parameter μ. L_4 and L_5 represent equilibria where all three bodies sit at the vertices of an equilateral triangle. For values of $\mu > \mu_1 \simeq 0.0385208$, both these equilibria are unstable. For $\mu < \mu_1$, ALFRIEND [1, 2] showed that the triangular points are unstable when $\mu = \mu_2$ and μ_3, special mass ratios which cause the linearized frequencies to be in the ratio of 1:2 and 1:3, respectively. For other values of $\mu < \mu_1$, stability of L_4 (and L_5) can be obtained by using Arnold's theorem. This was first done by DEPRIT and DEPRIT-BARTHOLOME [9], who calculated D_2 by hand. The value they obtained,

$$D_2 = -\frac{36 - 541\omega_1^2\omega_2^2 + 644\omega_1^4\omega_2^4}{8(1 - 4\omega_1^2\omega_2^2)(4 - 25\omega_1^2\omega_2^2)} \tag{88}$$

is non-zero for all values of μ except for $\mu = \mu_c \simeq 0.0109136$. For $\mu = \mu_c$, Arnold's theorem requires the quantity D_4 be found. This computation was performed by MEYER and SCHMIDT, who found $D_4 \simeq -66.6$. The non-zero value of D_4 implies stability, by Arnold's theorem.

In what follows we shall apply the results obtained in this paper to confirm the previous computations of DEPRIT and DEPRIT-BARTHOLOME [9] and MEYER and SCHMIDT [16].

The Hamiltonian for the circular restricted three-body problem about the equilibrium L_4 is:

$$H = \frac{1}{2}(P_1^2 + P_2^2) + P_1Q_2 - P_2Q_1 - \frac{(1-2\mu)}{2}Q_1 - \frac{\sqrt{3}}{2}Q_2 - \left(\frac{\mu}{\varrho_1} + \frac{1-\mu}{\varrho_2}\right), \tag{89}$$

where $\varrho_1^2 = Q_1^2 + Q_2^2 - Q_1 + \sqrt{3}\,Q_2 + 1$, $\varrho_2^2 = Q_1^2 + Q_2^2 + Q_1 + \sqrt{3}\,Q_2 + 1$. Expanding in a Taylor series about the origin, H becomes $\sum H_n$ where H_n contains terms of order $n + 2$ and H_0 is given by:

$$H_0 = \frac{1}{2}(P_1^2 + P_2^2) + P_1Q_2 - P_2Q_1 + \frac{1}{8}Q_1^2 - \frac{5}{8}Q_2^2 - \frac{3\sqrt{3}}{4}(1-2\mu)\,Q_1Q_2. \tag{90}$$

Then using the linearized differential equations corresponding to H_0, the characteristic equation for the system is found to be:

$$\lambda^4 + \lambda^2 + \tfrac{27}{16}(1-\gamma^2) = 0 \quad \text{where} \quad \gamma = 1 - 2\mu\,. \tag{91}$$

The eigenvalues λ have positive real parts for $\mu > \mu_1 = \dfrac{1}{2}\left(1 - \dfrac{\sqrt{69}}{9}\right)$ implying the system is linearly unstable. For $\mu < \mu_1$, the system is critically stable having eigenvalues $\pm i\omega_1$ and $\pm i\omega_2$ where:

$$0 < \omega_2 < \frac{\sqrt{2}}{2} < \omega_1 < 1\,, \qquad \omega_1^2 + \omega_2^2 = 1\,, \qquad \omega_1^2\omega_2^2 = \frac{27}{16}(1-\gamma^2)\,.$$

Using a canonical linear transformation (see [6]), $H_0(Q_m, P_m)$ is transformed into $H_0(q_m, p_m)$ which is of the form (30). Then following equations (30)—(34) we introduce the variables (x_m, y_m) and find the following components of K_2:

$$K_{2200} = \frac{-\omega_2^2(124\omega_1^4 - 696\omega_1^2 + 81)}{144(2\omega_1^2 - 1)^2(\omega_2^2 - 4\omega_1^2)}, \qquad K_{1111} = \frac{\omega_1\omega_2(64\omega_1^4 - 64\omega_1^2 - 43)}{6(2\omega_1^2 - 1)^2(\omega_2^2 - 4\omega_1^2)(\omega_1^2 - 4\omega_2^2)}, \tag{92) (93}$$

$$K_{0022} = \frac{\omega_1^2(124\omega_1^4 + 448\omega_1^2 - 491)}{144(2\omega_1^2 - 1)^2(4\omega_2^2 - \omega_1^2)} \tag{94}$$

and the first stability condition becomes:

$$D_2 = \frac{-644\omega_1^8 + 1288\omega_1^6 - 1185\omega_1^4 + 541\omega_1^2 - 36}{8(2\omega_1^2 - 1)^2(\omega_2 - 2\omega_1)(\omega_2 + 2\omega_1)(2\omega_2 - \omega_1)(2\omega_2 + \omega_1)}$$

which is equivalent to the expression (88) found by DEPRIT and DEPRIT-BARTHOLOME [9]. Then, on $0 < \mu < \mu_1$, $D_2 = 0$ only for:

$$\mu = \mu_c = \frac{3\sqrt{483} - \sqrt{2}\sqrt{\sqrt{199945} + 3265}}{6\sqrt{483}} \simeq 0.0109136\,.$$

At this value of $\mu = \mu_c$, the components of K_4 become:

$$K_{3300} \simeq 0.219259187i + 6.52\text{E}{-37}, \qquad K_{2211} \simeq -7.79324843i + 3.74\text{E} - 35\,, \tag{95), (96}$$

$$K_{1122} \simeq 209.933620i + 2.35\text{E}{-34}\,, \qquad K_{0033} \simeq 14.5264460i + 1.75\text{E}{-34} \tag{97), (98}$$

and D_4 becomes:

$$D_4 \simeq -66.6 - 4.27\text{E}{-36}i\,. \tag{99}$$

The very small real part of each K_{jlrs} and imaginary part of D_4 results from taking only a finite number of digits (40 in fact) in the numerical approximation. Because the real part is so much larger than the error term, the approximation $D_4 \simeq -66.6$ is accurate and $D_4 \neq 0$. Hence, at $\mu = \mu_c$, the Hamiltonian system is stable. These values for the coefficients of K_4 and D_4 agree with those obtained by MEYER and SCHMIDT [16].

Acknowledgement: The authors would like to thank Dr. DIETER ARMBRUSTER for his helpful advice with MACSYMA. This work was partially supported by grants from the National Science Foundation, AFOSR, and ARO through the Mathematical Sciences Institute at Cornell.

References

1 ALFRIED, K. T.: The stability of the triangular Lagrangian points for commensurability of order two. Celest. Mech. 1 (1970), 351–359.
2 ALFRIEND, K. T.: The stability of the triangular Lagrangian points for commensurability of order three. Celest. Mech. 4 (1971), 60–77.
3 ARNOLD, V. I.: Mathematical methods of classical mechanics. Graduate Texts in Mathematics, Vol. 60, Springer-Verlag, NY 1978.
4 ARNOLD, V. I.: The stability of the equilibrium positions of a Hamiltonian system of ordinary differential equations in the general elliptic case. Soviet Math. Doklady 2 (1961), 247–249.
5 BIRKHOFF, G.: Dynamical systems. Amer. Math. Soc., Providence, RI, 1966.
6 BREAKWELL, J.; PRINGLE, R.: Resonances affecting motion near the Earth-Moon equilateral libration points. In: Progress in astronautics and aeronautics: Methods in astrodynamics and celestial mechanics. Vol. 17, Academic Press, NY 1966.
7 COPPOLA, V. T.: M. S. Thesis, Cornell University 1988.
8 DEPRIT, A.: Canonical transformations depending on a parameter. Celestial mech. 1 (1969), 1–31.
9 DEPRIT, A.; DEPRIT-BARTHOLOME, A.: Stability of the triangular Lagrangian points. Astronom. J. 72 (1967) 2, 173–179.
10 DRAGT, A.; FINN, J. M.: Lie series and invariant functions for analytic symplectic maps. J. Math. Phys. 17 (1976), 2215.
11 GOLDSTEIN, H.: Classical mechanics. 2nd ed., Addison-Wesley, Reading, Mass. 1980.
12 HORI, G.: Theory of general perturbations with unspecified canonical variables. Publ. Astron. Soc. Japan 18 (1966) 4, 287–296.
13 LEN, J.: Nonlinear parametric excitation with averaging and Lie transform methods. Ph. d. Thesis, Cornell University 1987.
14 LEN, J.; RAND, R. H.: Lie transforms applied to a nonlinear parametric excitation problem. Internat. J. Nonlinear mech. 23 (1988), 297–313.
15 LICHTENBERG, A. J.; LIEBERMAN, M. A.: Regular and stochastic motion. Appl. Math. Sci. Vol. 38, Springer-Verlag, NY 1983.
16 MEYER, K. R.; SCHMIDT, D. S.: The stability of the Lagrange triangular point and a theorem of Arnold. J. Differential Equations 62 (1986) 2, 222–236.
17 MONTH, L.; RAND, R. H.: Bifurcations of 4:1 subharmonics in the nonlinear Mathieu equation. Mechanics Res. Commun. 9 (1982) 4, 233–240.
18 RAND, R. H.: Computer algebra in applied mathematics: An introduction to MACSYMA. Research Notes in Mathematics, No. 94, Pitman Publishing, Boston 1984.
19 RAND, R. H.; ARMBRUSTER, D.: Perturbation methods, bifurcation theory and computer algebra. Appl. Math. Sci. Vol. 65, Springer-Verlag, NY 1987.
20 SZEBEHELY, V.: Theory of orbits: The restricted problem of three bodies. Academic Press, New York 1967.

Received May 2, 1988

Address: Dr. RICHARD H. RAND, Professor; Dr. VINCENT T. COPPOLA, Department of Theoretical and Applied Mechanics College of Engineering, Cornell University, Thurston Hall, Ithaca, NY 14853–1503, USA