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A COMPREHENSIVE APPROACH TO THE FREE VIBRATION ANALYSIS OF RECTANGULAR PLATES BY USE OF THE METHOD OF SUPERPOSITION

1. INTRODUCTION

There are few topics in mechanics which have received more attention in the literature than the free vibration of rectangular plates. In spite of the great effort which has gone into this area it remains a fact that, with the exception of plates with at least two opposite edges simply supported, the treatment of this subject up to this time is far from satisfactory. Most of the solutions put forward are either approximate in that they do not satisfy completely the boundary conditions, or the governing differential equation, or they involve complicated finite difference or energy methods.

The authors are currently engaged in an interesting study of these classical vibration problems in which they are exploiting the method of superposition. It is found that solutions of any desired degree of exactitude are readily obtained for any combination of free, simply supported, and clamped edge conditions. Modal shape information is immediately available in terms of the familiar trigonometric and hyperbolic functions. An attractive feature of this method of solution is that unlike many of the more complicated finite difference and other numerical methods, it is immediately comprehensible to the practicing engineer. It will become obvious that any boundary conditions, including elastic restraints at the edges, are easily handled. The purpose of this note is to briefly describe the steps the authors are employing to utilize this method.

2. PLATES WITH COMBINATIONS OF CLAMPED-SIMPLY SUPPORTED EDGES

Only one basic building block is required to solve all of the problems in this family. For a plate with simple support along all edges and a distributed bending moment along the edge $\eta = 1$ (see Figure 1) the authors have shown that solution to the plate dynamic equilibrium equation may be expressed as

$$W(\xi,\eta) = \sum_{m=1}^{k^*} \frac{E_m}{(\beta_m^2+\gamma_m^2)} \left\{ \frac{\sin\gamma_m\eta}{\sin\gamma_m} - \frac{\sinh\beta_m\eta}{\sinh\beta_m} \right\} \sin m\pi\xi + \\ + \sum_{m=k^*+1}^{k} \frac{E_m}{(\beta_m^2-\gamma_m^2)} \left\{ \frac{\sinh\gamma_m\eta}{\sinh\gamma_m} - \frac{\sinh\beta_m\eta}{\sinh\beta_m} \right\} \sin m\pi\xi,$$

where the first summation pertains to terms for which $\lambda^2 > (m\pi)^2$ and the second summation pertains to terms for which $\lambda^2 < (m\pi)^2$ *(see* the list of nomenclature in the Appendix) and the distributed bending moment is expressed as

$$Mb^2/Da = \sum_{m=0}^{k} E_m \sin m\pi\xi.$$

Solution to the rectangular plate vibration problem with all edges clamped is obtained by superimposing four solutions of the above type which are essentially identical except that through interchange of variables each solution has a bending moment distribution along a different edge. Requirement that the slope must vanish at all edges gives rise to $4k$ simul-

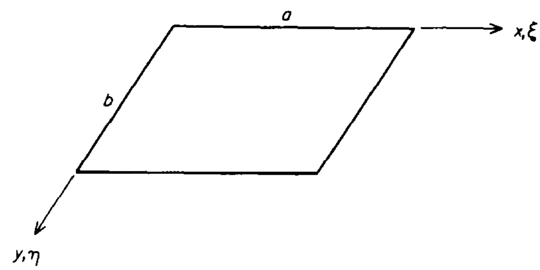

Figure 1. Co-ordinate system employed.

taneous linear algebraic homogeneous equations for the $4k$ Fourier coefficients. Frequency eigenvalues are, of course, obtained by incrementing λ^2, and hence β_m, etc., to find those values of λ^2 which permit a non-trivial solution for the Fourier coefficients. Setting one of the non-zero coefficients equal to unity permits solution of the algebraic equations for the other coefficients and hence a summation of analytical functions for the modal shape associated with any frequency. The aspect ratio can be varied as desired.

3. PLATES WITH ONE OR MORE FREE EDGES

It is known that there are numerous solutions which can be combined to give the desired solutions. The object of the skilled analyst is to choose the most simple of possible combinations. For plates with not more than three free edges the authors have combined solutions which are similar to the one discussed above except that the edges $\eta = 1$ and $\eta = 0$ are simply supported or free with distributed bending moment as required and the sine functions above are replaced with functions of the type $\sin m\pi\xi/2$, where m takes on odd integer values. It will be appreciated that the edge, $\xi = 1$, of this solution is free of vertical edge reaction but has sufficient bending moment distributed along it to eliminate slope. This bending moment can easily be eliminated by a proper combination of these latter solutions and properly constraining the Fourier coefficients. For plates with all free edges the only change necessary is to replace the sine functions immediately above with functions of the type $\cos m\pi\xi$ where m can take on all positive integers. The edges at $\xi = 0$ and $\xi = 1$ are now free of vertical edge reaction and existing edge bending moments can be eliminated by a proper superposition of solutions.

4. SUMMARY

The authors have found the above techniques to constitute a powerful means for solving rectangular plate problems. At the time of writing, solutions for plates with two adjacent simply supported edges and two adjacent free edges have been obtained. The first 20 eigenvalues for plates with all edges clamped have also been determined for a full range of aspect ratio and they are shown to be accurate to within less than one half of one percent. It will be appreciated that solutions for any combination of clamped-simply supported edge conditions can easily be obtained from the all-clamped solution by simply deleting appropriate solutions from the all-clamped combination. In Figure 2 contour lines for first mode vibration of a plate with two adjacent clamped and two adjacent simply supported edges is presented. The higher density of the contour lines along the simply supported edges will be noted.

The method of superposition is currently being used by the authors to good advantage to obtain solutions of any desired degree of accuracy to all of the problems discussed. It is found to be easily utilized and unlike more complicated methods is readily comprehensible to the practicing engineer. Eigenvalues for all modes, aspect ratios, and boundary conditions are

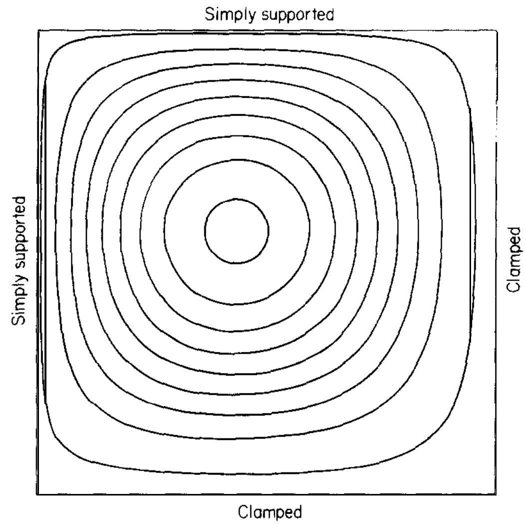

Figure 2. View of computed contour lines for first mode vibration of square plate with two adjacent edges clamped and the other two simply supported.

readily obtained. Modal shapes are expressed in terms of familiar analytic functions. Results of these studies will be made available in future publications.

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APPENDIX : NOMENCLATURE

- a plate dimension in x direction
- b plate dimension in y direction
- D plate flexural rigidity, $= (Eh^3/12)(1-\nu^2)$
- E modulus of elasticity of material
- h plate thickness
- k number of terms in Fourier expansions
- M bending moment per unit length along plate edge
- m positive integer
- w plate lateral deflection
- x,y plate spatial co-ordinates
- ξ dimensionless plate spatial co-ordinate x/a
- η dimensionless plate spatial co-ordinate y/b
- ρ mass per unit area of plate
- ω circular frequency of vibration
- ϕ plate aspect ratio, $= b/a$
- λ^2 $= \omega a^2 \sqrt{\rho/D}$
- μ_m $= m\pi$
- β_m $= \phi\sqrt{\lambda^2 + \mu_m^2}$
- γ_m $= \phi\sqrt{\lambda^2 - \mu_m^2}$ or $\phi\sqrt{\mu_m^2 - \lambda^2}$, whichever is real
- ν Poisson's ratio of material