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LETTERS TO THE EDITOR

IN-PLANE VIBRATION OF POINT-SUPPORTED RECTANGULAR PLATES

The natural frequencies (the dimensionless frequency parameters) and the mode shapes of in-plane vibration are presented for point-supported rectangular plates. For this purpose, the deflection displacements of the plate are written in a series of the product of the power functions, and the frequency equation is derived by the Ritz method. There have been many papers available on in-plane vibration of circular plates, but few about rectangular plates. Irie *et al.* [1] studied annular plates by using the transfer matrix method and reviewed some papers dealing with in-plane vibration of circular plates. Kawai and Yoshimura [2] and Yamabuchi and Kagawa [3] investigated the in-plane vibration of rectangular plates by using the finite element method.

Figure 1 shows a rectangular plate supported at several points. With the length of two edges denoted by $2a$, $2b$ and the co-ordinates of the supports by (x_i, y_i), the rectangular co-ordinates (x, y) are taken in the middle surface of the plate. The maximum kinetic and strain energies of the plate are written as

$$T=\tfrac{1}{2}\rho H\omega^2\int_{-b}^{b}\int_{-a}^{a}(u^2+v^2)\,\mathrm{d}x\,\mathrm{d}y, \tag{1}$$

$$U=\frac{D}{2}\int_{-b}^{b}\int_{-a}^{a}\left\{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2-2(1-\nu)\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{1-\nu}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right\}\mathrm{d}x\,\mathrm{d}y, \tag{2}$$

in terms of the maximum in-plane deflection displacements u, v in the x- and y-directions, respectively, where ρ is the mass density, H is the plate thickness, ω is the frequency in radians/second, and D is the extensional rigidity expressed as $D=EH/(1-\nu^2)$ by using the Young's modulus E, the Poisson ratio ν and the plate thickness. Introducing the non-dimensional variables $\xi=x/a$ and $\eta=y/b$ to equations (1) and (2) enables the deflection displacements of the plate to be expressed as

$$u=\sum_{m=0}^{M}\sum_{n=0}^{N}A_{mn}\xi^m\eta^n,\qquad v=\sum_{m=0}^{M}\sum_{n=0}^{N}B_{mn}\xi^m\eta^n, \tag{3}$$

in terms of polynomial functions. The Lagrange functional of the point-supported plates is written as

$$L=T-U-\sum_{i=1}^{L}\{\lambda_i u(\xi_i,\eta_i)+\mu_i v(\xi_i,\eta_i)\}, \tag{4}$$

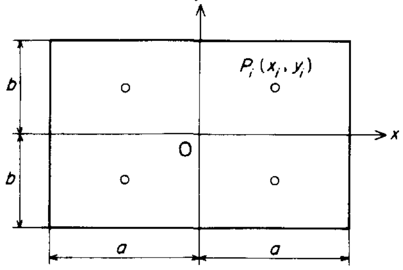

Figure 1. A point-supported rectangular plate.

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where λ_i and μ_i are the Lagrange multipliers expressing the components of the unknown reaction forces at the point-supports. By substituting the equations which are obtained by the substitution of equations (3) into equations (1) and (2) into the conditions for a stationary value of the Lagrange functional (4),

$$\partial L/\partial A_{kl}=0, \quad \partial L/\partial B_{kl}=0, \qquad (k=0,1,2,\ldots,M;\ l=0,1,2,\ldots,N),$$
$$\partial L/\partial \lambda_j=0, \quad \partial L/\partial \mu_j=0, \qquad (j=1,2,\ldots,I), \tag{5}$$

the following equations are derived:

$$\sum_{m=0}^{M}\sum_{n=0}^{N}\left\{\beta\phi_{mk}^{(11)}\phi_{nl}^{(00)}+\frac{1-\nu}{2\beta}\phi_{mk}^{(00)}\phi_{nl}^{(11)}-\Omega^2\phi_{mk}^{(00)}\phi_{nl}^{(00)}\right\}A_{mn}$$
$$+\sum_{m=0}^{M}\sum_{n=0}^{N}\left\{\nu\phi_{mk}^{(01)}\phi_{nl}^{(10)}+\frac{1-\nu}{2}\phi_{mk}^{(10)}\phi_{nl}^{(01)}\right\}B_{mn}+\sum_{i=1}^{I}\lambda_i\xi_i^k\eta_i^l=0, \tag{6}$$

$$\sum_{m=0}^{M}\sum_{n=0}^{N}\left\{\nu\phi_{mk}^{(10)}\phi_{nl}^{(01)}+\frac{1+\nu}{2}\phi_{mk}^{(01)}\phi_{nl}^{(10)}\right\}A_{mn}$$
$$+\sum_{m=0}^{M}\sum_{n=0}^{N}\left\{\frac{1}{\beta}\phi_{mk}^{(00)}\phi_{nl}^{(11)}+\frac{1-\nu}{2}\beta\phi_{mk}^{(11)}\phi_{nl}^{(00)}-\Omega^2\phi_{mk}^{(00)}\phi_{ml}^{(00)}\right\}B_{mn}+\sum_{i=1}^{I}\mu_i\xi_i^k\eta_i^l=0, \tag{7}$$

$$\sum_{m=0}^{M}\sum_{n=0}^{N}A_{mn}\xi_j^m\eta_j^n=0, \qquad \sum_{m=0}^{M}\sum_{n=0}^{N}B_{mn}\xi_j^m\eta_j^n=0. \tag{8}$$

Here

$$\phi_{mk}^{(ij)}=\int_{-1}^{1}\left(\frac{\mathrm{d}^{(i)}\xi^m}{\mathrm{d}\xi^{(i)}}\right)\left(\frac{\mathrm{d}^{(j)}\xi^k}{\mathrm{d}\xi^{(j)}}\right)\mathrm{d}\xi, \qquad \phi_{nl}^{(ij)}=\int_{-1}^{1}\left(\frac{\mathrm{d}^{(i)}\eta^n}{\mathrm{d}\eta^{(i)}}\right)\left(\frac{\mathrm{d}^{(j)}\eta^l}{\mathrm{d}\eta^{(j)}}\right)\mathrm{d}\eta. \tag{9}$$

For simplicity of the analysis, the following dimensionless quantities have been introduced:

$$\beta=b/a, \qquad \Omega^2=\rho H\omega^2 ab/D \quad \text{(frequency parameter)}. \tag{10}$$

Equations (6), (7) and (8) are linear homogeneous equations in the unknown coefficients A_{mn}, B_{mn}, λ_i and μ_i. The natural frequencies of the plate are determined by calculating the eigenvalues Ω, and the mode shapes of vibration are determined from equations (3) by calculating the coefficients corresponding to the eigenvalues.

Consider a plate supported at four points symmetrically located with respect to two axes. In a plate with structural symmetry, only symmetrical (S-type) and antisymmetrical

TABLE 1

Power of deflection functions

Vibration mode	u: m	u: n	v: m	v: n
SS	Odd	Even	Even	Odd
SA	Even	Even	Odd	Odd
AS	Odd	Odd	Even	Even
AA	Even	Odd	Odd	Even

(A-type) vibrations arise due to the axes of symmetry, and it is sufficient to consider only a quarter-part of the plate. It is possible to classify the vibration modes into four types by taking the integer numbers m and n written in Table 1 as the powers in equations (3).

In the numerical calculation, the number of terms $M \times N$ of the series in equations (3) may be truncated at an appropriate finite number by considering the convergence of the numerical solution. Table 2 shows the convergence characteristics of the eigenvalues of vibration for a point-supported square plate when the number of term $M \times N$ increases. The values converge within the range of two or three significant figures with an increase in number of these terms. In the present calculation, 7×7 terms were used which assures sufficient accuracy for practice.

TABLE 2

Convergence characteristics of eigenvalues of vibration for a point-supported square plate: $\nu = 0{\cdot}3$, $\xi_1 = \eta_1 = 0{\cdot}5$

$M\times N$	SA-1	SS-1	SS-2	AA-1
5×5	1·610	1·489	2·020	2·002
6×6	1·593	1·489	1·987	1·962
7×7	1·469	1·488	1·919	1·921
8×8	1·430	1·488	1·877	1·865

The eigenvalues of vibration for a completely free rectangular plate obtained here are compared with the values obtained by Kawai and Yoshimura [2] in Table 3, and the values are in good agreement.

Figure 2 shows the eigenvalues of vibration Ω for square plates versus the co-ordinate $\xi_1 = \eta_1$ of a supported point. In the case of $\xi_1 = \eta_1 = 0$, the plate is supported by a point at the center, and the eigenvalues of vibration for SS- and AA-type vibrations agree with the values for completely free plates.

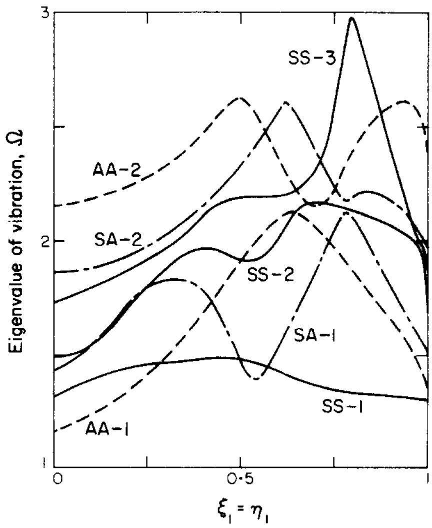

Figure 2. Eigenvalues of vibration of point-supported square plates. $\nu = 0{\cdot}3$; $\beta = 1$.

TABLE 3

Comparison of eigenvalues of vibration for a free rectangular plate; $\nu = 0{\cdot}3$, $\beta = 0{\cdot}5$

Vibration mode	Present	Kawai and Yoshimura [2]
AS-1	0·6907	0·7277
SS-1	1·047	1·049
AA-1	1·155	1·228
AA-2	1·671	1·809
AS-2	1·691	1·787
SS-2	1·840	1·900

Figure 3 shows the eigenvalues of vibration Ω versus the length ratio. In the case of $\beta = 1$, the eigenvalues of vibration for SA- and AS-type vibration take on the same value because of the symmetry on the diagonals of the plate.

Figure 4 shows the mode shapes of point-supported square plates. The points marked with ○ represent the locations of the supports. The thick lines show the composition of the x- and y-direction displacements, where the maximum displacements are taken to have unit values. With variation of the co-ordinate $\xi_1 = \eta_1$, the mode shapes change, being affected by the locations of the supports. The eigenvalues of vibration increase considerably when the plate is supported at points where the displacements are large.

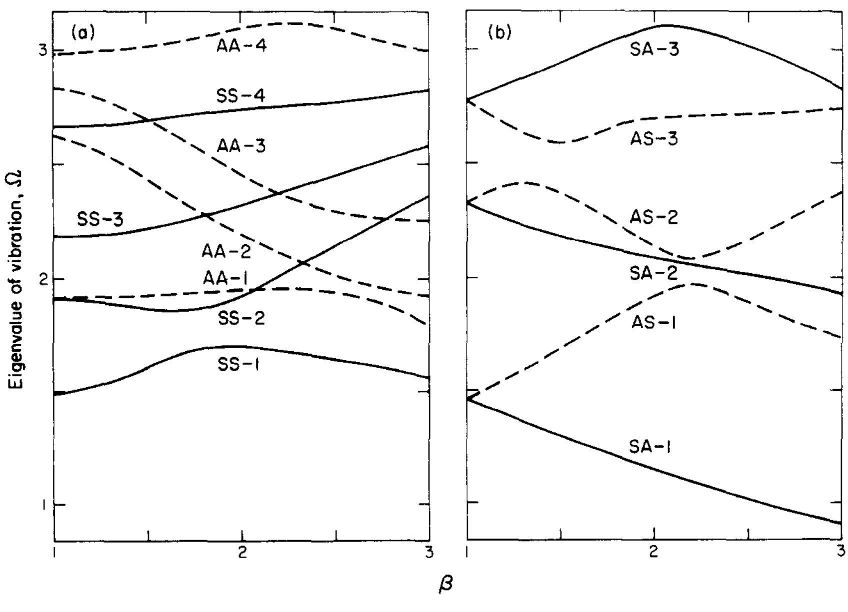

Figure 3. Eigenvalues of vibration of point-supported rectangular plates. $\nu = 0{\cdot}3$; $\xi_1 = \eta_1 = 0{\cdot}5$.

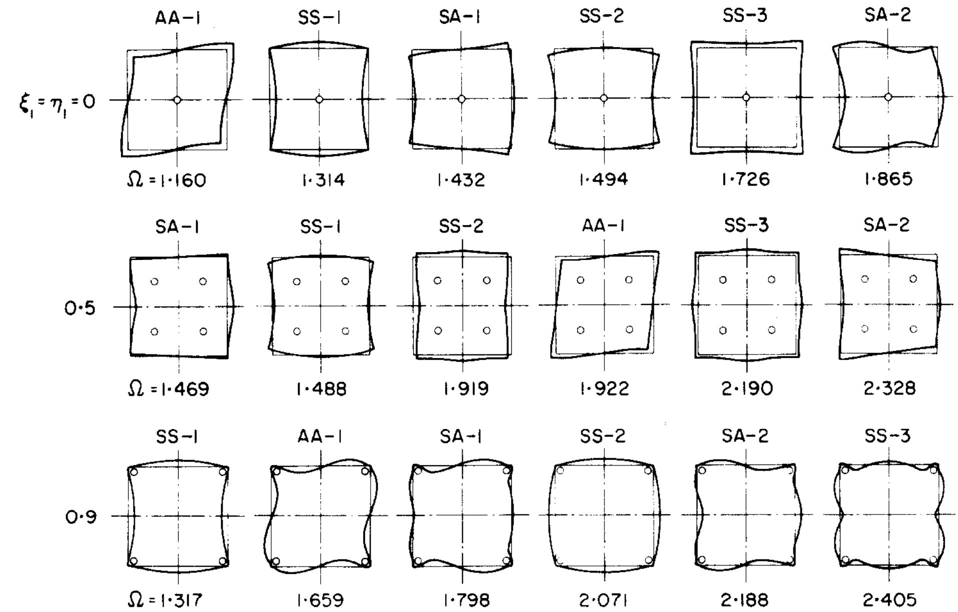

Figure 4. Mode shapes of point-supported square plates. $\nu = 0{\cdot}3$; $\beta = 1$.

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