Quaternions, Finite Rotation and Euler Parameters

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May 16, 2002

Son: Well, Papa, can you multiply triplets?
Father: No [sadly shaking his head], I can only add and subtract them.
(William Rowan Hamilton, Conversation with his sons (1843))

A quaternion is a collection of four real parameters, of which the first is considered as a scalar and the other three as a vector in three-dimensional space. In addition, the following operations are defined. If $q = (q_0, \mathbf{q}) = (q_0, q_1, q_2, q_3)$ and $p = (p_0, \mathbf{p}) = (p_0, p_1, p_2, p_3)$ are two quaternions, their sum is defined as

$$q + p = (q_0 + p_0, \mathbf{q} + \mathbf{p}), \tag{1}$$

and their product (non-commutative) as

$$q \circ p = (q_0 p_0 - \mathbf{q} \cdot \mathbf{p}, q_0 \mathbf{p} + p_0 \mathbf{q} + \mathbf{q} \times \mathbf{p}). \tag{2}$$

The adjoint quaternion of q is defined as $\overline{q} = (q_0, -\mathbf{q})$ and the length or norm as $|q| = \sqrt{(\overline{q} \circ q)_0} = \sqrt{q_0^2 + \mathbf{q} \cdot \mathbf{q}}$. Note that $|q \circ p| = |q||p|$. There are two special quaternions, the unit element $1 = (1, \mathbf{0})$ and the zero element $0 = (0, \mathbf{0})$. The reciprocal of a quaternion $q \neq 0$ is $q^{-1} = \overline{q}/|q|^2$. The quaternion with a norm of one, $|q| = 1$, is a unit quaternion.

If a quaternion is looked upon as a four-dimensional vector, the quaternion product can be described by a matrix-vector product as

$$\begin{aligned} q \circ p &= \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & q_0 \mathbf{I}_3 + \tilde{\mathbf{q}} \end{pmatrix} \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} = \mathbf{Q} \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix}, \\ p \circ q &= \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & q_0 \mathbf{I}_3 - \tilde{\mathbf{q}} \end{pmatrix} \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} = \overline{\mathbf{Q}} \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix}. \end{aligned} \tag{3}$$

Here we have used the tilde notation for the antisymmetric matrix $\tilde{\mathbf{q}}$ from the vector $\mathbf{q}$, which is defined by the matrix-vector notation for the vector cross product $\mathbf{q} \times \mathbf{x} = \tilde{\mathbf{q}}\mathbf{x}$. This skew-symmetric matrix is

$$\tilde{\mathbf{q}} = \begin{pmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{pmatrix}. \tag{4}$$

The quaternion matrices $\mathbf{Q}$ and $\overline{\mathbf{P}}$ commute, $\mathbf{Q}\overline{\mathbf{P}} = \overline{\mathbf{P}}\mathbf{Q}$. The matrices of the adjoint quaternion $\overline{q}$ are $\mathbf{Q}^T$ and $\overline{\mathbf{Q}}^T$.

If we associate the quaternion $x' = (0, \mathbf{x}')$ with the three-dimensional vector $\mathbf{x}'$ and define the operation, with the unit quaternion q, as

$$x = q \circ x' \circ q^{-1} = q \circ x' \circ \overline{q}, \tag{5}$$

then this transformation, from x' to x, represents a rotation. The resulting quaternion x is a vectorial quaternion with the same length as x'. The case of reflection, the other possibility, can be excluded. The rotation matrix $\mathbf{R}$ in terms of the unit quaternions q can be derived from equation (5) as

$$\mathbf{x} = (q_0^2 - \mathbf{q}\cdot\mathbf{q})\mathbf{x}' + 2q_0(\mathbf{q}\times\mathbf{x}') + 2(\mathbf{q}\cdot\mathbf{x}')\mathbf{q} = \mathbf{R}\mathbf{x}' \tag{6}$$

with

$$\mathbf{R} = \begin{pmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_2q_1+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_3q_1-q_0q_2) & 2(q_3q_2+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{pmatrix}. \tag{7}$$

This rotation matrix can also be written with the help of the quaternion matrix representation according to

$$\begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R} \end{pmatrix} = \mathbf{Q}\overline{\mathbf{Q}}^T = \overline{\mathbf{Q}}^T\mathbf{Q}. \tag{8}$$

The quaternion q in the rotation matrix $\mathbf{R}$ according to equation (7), is identified as the set of Euler parameters for the description of finite rotation. According to Euler's theorem on finite rotation, a rotation in space can always be described by a rotation along a certain axis over a certain angle. With the unit vector $\mathbf{e}_\mu$ representing the axis and the angle of rotation μ, right-handed positive, the Euler parameters q can be interpreted as

$$q_0 = \cos(\mu/2) \quad \text{and} \quad \mathbf{q} = \sin(\mu/2)\mathbf{e}_\mu. \tag{9}$$

Since the Euler parameters are unit quaternions the subsidiary condition,

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1, \tag{10}$$

must always be satisfied. The quaternion x' in (5) can now be associated with the algebraic components of a vector in a body fixed frame and the quaternion x as the corresponding components expressed in a space fixed frame.

The Euler parameters for successive rotation are given by the quaternion product of the Euler parameters describing the individual rotations.

Before we derive the rotational equations of motion for a spatial rigid body in terms of Euler parameters we have to express the angular velocities and accelerations in terms of the Euler parameters and its time derivatives. By differentiation of the rotational transformation (5) as in

$$\dot{x} = \dot{q} \circ x' \circ \overline{q} + q \circ x' \circ \dot{\overline{q}}, \tag{11}$$

and substitution of the body fixed coordinates according to $x' = \overline{q} \circ x \circ q$, realizing that $\overline{q} \circ q$ is the unit element $(1, \mathbf{0})$, the velocity reads

$$\dot{x} = \dot{q} \circ \overline{q} \circ x + x \circ q \circ \dot{\overline{q}}. \tag{12}$$

The scalar part of the products $\dot{q} \circ \overline{q}$ and $q \circ \dot{\overline{q}}$ are zero, since q is a unit quaternion, and the vector parts are opposite so we may write: $\dot{q} \circ \overline{q} = (0, \mathbf{w})$ and $q \circ \dot{\overline{q}} = (0, -\mathbf{w})$. The velocity $\dot{x}$ now has a zero scalar part, as expected, and a vectorial part, $\dot{\mathbf{x}} = 2\mathbf{w} \times \mathbf{x}$, so $\boldsymbol{\omega} = 2\mathbf{w}$. We conclude that the angular velocity $\boldsymbol{\omega}$ expressed in the space fixed reference in terms of the Euler parameters q and its time derivatives is given by

$$\omega = 2\dot{q} \circ \overline{q} \quad \text{or} \quad \begin{pmatrix} 0 \\ \boldsymbol{\omega} \end{pmatrix} = 2\overline{\mathbf{Q}}^T \begin{pmatrix} \dot{q}_0 \\ \dot{\mathbf{q}} \end{pmatrix}. \tag{13}$$

The inverse, the time derivatives $\dot{q}$ of the Euler parameters for given q and ω, can be found as

$$\dot{q} = \frac{1}{2}\omega \circ q \quad \text{or} \quad \begin{pmatrix} \dot{q}_0 \\ \dot{\mathbf{q}} \end{pmatrix} = \frac{1}{2}\overline{\mathbf{Q}} \begin{pmatrix} 0 \\ \boldsymbol{\omega} \end{pmatrix}. \tag{14}$$

Note that these time derivatives are always uniquely defined, opposed to the classical combination of 3 parameters for describing spatial rotation as in for example Euler angles, Rodrigues parameters or Cardan angles. The angular velocities $\boldsymbol{\omega}'$ expressed in a body fixed reference frame can be derived in the same manner, or by application of the rotational transformation (8), as

$$\omega' = 2\overline{q} \circ \dot{q} \quad \text{or} \quad \begin{pmatrix} 0 \\ \boldsymbol{\omega}' \end{pmatrix} = 2\mathbf{Q}^T \begin{pmatrix} \dot{q}_0 \\ \dot{\mathbf{q}} \end{pmatrix}, \tag{15}$$

and with the inverse

$$\dot{q} = \frac{1}{2}q \circ \omega' \quad \text{or} \quad \begin{pmatrix} \dot{q}_0 \\ \dot{\mathbf{q}} \end{pmatrix} = \frac{1}{2}\mathbf{Q} \begin{pmatrix} 0 \\ \boldsymbol{\omega}' \end{pmatrix}. \tag{16}$$

The angular accelarations are found by differentiation of the expressions for ω and ω', resulting in

$$\begin{pmatrix} 0 \\ \dot{\boldsymbol{\omega}} \end{pmatrix} = 2\overline{\mathbf{Q}}^T \begin{pmatrix} \ddot{q}_0 \\ \ddot{\mathbf{q}} \end{pmatrix} + 2 \begin{pmatrix} |\dot{q}|^2 \\ \mathbf{0} \end{pmatrix}, \tag{17}$$

and expressed in the body fixed reference frame

$$\begin{pmatrix} 0 \\ \dot{\boldsymbol{\omega}}' \end{pmatrix} = 2\mathbf{Q}^T \begin{pmatrix} \ddot{q}_0 \\ \ddot{\mathbf{q}} \end{pmatrix} + 2 \begin{pmatrix} |\dot{q}|^2 \\ \mathbf{0} \end{pmatrix}. \tag{18}$$

The inverse, the second order time derivatives $\ddot{q}$ of the Euler parameters in terms of q, $\dot{q}$ and $\dot{\boldsymbol{\omega}}$, goes without saying. The equations of motion for the rotation of a rigid body in a space with the components of the inertia tensor as matrix $\mathbf{J}'$

and the vector of applied torques $\mathbf{M}'$, all at the centre of mass expressed in the body fixed frame, are

$$\mathbf{J}'\dot{\boldsymbol{\omega}}' = \mathbf{M}' - \boldsymbol{\omega}' \times (\mathbf{J}'\boldsymbol{\omega}'), \tag{19}$$

They can be expressed in terms of Euler parameters and its time derivatives by application of the principle of virtual power and introduction of the Lagrangian multiplier λ for the norm constraint (10) written as

$$\varepsilon_q = q_0^2 + q_1^2 + q_2^2 + q_3^2 - 1 = 0, \tag{20}$$

resulting in the virtual power equation for a rigid body as

$$(\mathbf{M}' - \mathbf{J}'\dot{\boldsymbol{\omega}}' - \boldsymbol{\omega}' \times (\mathbf{J}'\boldsymbol{\omega}'))^T \delta\boldsymbol{\omega}' = \lambda \delta\dot{\varepsilon}_q. \tag{21}$$

The virtual constraint rate can be derived from (20) as

$$\delta\dot{\varepsilon}_q = 2q_0\delta\dot{q}_0 + 2\mathbf{q}^T\delta\dot{\mathbf{q}}. \tag{22}$$

Substitution of these expressions and the expressions for the angular velocities (15) and the angular accelerations (18) in the virtual power equation (21) and taking arbitrary virtual Euler parameter velocities yields after adding the constraints on the accelerations of the Euler parameters as in (17) or (18), the equations of motion for a rigid body expressed in terms of Euler parameters as

$$\left[\begin{array}{cc} 4\mathbf{Q}\begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{J}' \end{pmatrix}\mathbf{Q}^T & 2\begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix} \\ 2(q_0, \mathbf{q}^T) & 0 \end{array}\right] \left[\begin{array}{c} \ddot{q}_0 \\ \ddot{\mathbf{q}} \\ \lambda \end{array}\right] = \left[\begin{array}{c} 2\mathbf{Q}\begin{pmatrix} 0 \\ \mathbf{M}' \end{pmatrix} + 8\dot{\mathbf{Q}}\begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{J}' \end{pmatrix}\dot{\mathbf{Q}}^T\begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix} \\ -2|\dot{q}|^2 \end{array}\right]. \tag{23}$$

The multiplier λ can for this single body be obtained by premultiplying the first four equations by $(q_0, \mathbf{q})^T$ and is indentified as twice the rotational kinetic energy of the body

$$\lambda = 4\begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix}^T \dot{\mathbf{Q}}\begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{J}' \end{pmatrix}\dot{\mathbf{Q}}^T\begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix} = \boldsymbol{\omega}'^T\mathbf{J}'\boldsymbol{\omega}'. \tag{24}$$

The transformations of an applied torque, body fixed $\mathbf{M}'$ or space fixed $\mathbf{M}$, to the torque parameters $(f_0, \mathbf{f})$, which are dual to the Euler parameters, are apparently

$$\begin{pmatrix} f_0 \\ \mathbf{f} \end{pmatrix} = 2\mathbf{Q}\begin{pmatrix} 0 \\ \mathbf{M}' \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} f_0 \\ \mathbf{f} \end{pmatrix} = 2\overline{\mathbf{Q}}\begin{pmatrix} 0 \\ \mathbf{M} \end{pmatrix}. \tag{25}$$