Multibody Dynamics A

wb1310

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Orientation of a rigid body in space.

We can keep track of any point on the rigid body by means of translation $\mathbf{t}$ of a point of the body and and a rotation about $\mathbf{t}$ described by means of an orthogonal rotation matrix $\mathbf{R}$.

An arbitrary location $\mathbf{p}'$ in the rigid body expressed in the body fixed coordinate system (x', y', z') can be exprressed in the space fixed coordinate system (x, y, z) (f.i. inertia system):

$$\mathbf{p} = \mathbf{t} + \mathbf{R}\mathbf{p}'$$

The translation vector $\mathbf{t}$ has three components (t_x, t_y, t_z). The orthogonal rotation matrix $\mathbf{R}$ has 9 components but can be described by means of 3 independent parameters due to the orthogonality conditions:
$\mathbf{R}\mathbf{R}^T = \mathbf{I}$ which are 6 (due to symmetry) in total.

An example of this parametrization are:
Euler angles, but there are many more see f.i. Wittenburg (1977).

Euler angles (φ, χ, ψ), also known as 3-1-3 or z-x-z angles, as a parameterizations of $\mathbf{R}$.

How do we get $\mathbf{R}$ out of (φ, χ, ψ) ?

Recipe:

- 1 Rotate with φ about the z-axis
- 2 Rotate with χ about the rotated x-axis
- 3 Rotate with ψ about the rotaated z-axis

Now you can think of this rotation matrix $\mathbf{R}$ as the product of 3 successive rotation matrices:

$\mathbf{R} = \mathbf{R}_\varphi \mathbf{R}_\chi \mathbf{R}_\psi$

(Note the order!)

The individual matrices are:

Rotate with φ about the z-axis

$$\mathbf{R}_\varphi = \begin{pmatrix} c_\varphi & -s_\varphi & 0 \\ s_\varphi & c_\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(with: $s_x = \sin x$ en $c_x = \cos x$)

Rotate with χ about the rotated x-axis

$$\mathbf{R}_\chi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\chi & -s_\chi \\ 0 & s_\chi & c_\chi \end{pmatrix}$$

Rotate with ψ about the rotaated z-axis

$$\mathbf{R}_\psi = \begin{pmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The rotation matrix $\mathbf{R} = \mathbf{R}_\varphi \mathbf{R}_\chi \mathbf{R}_\psi$ is given by:

$$\mathbf{R} = \begin{pmatrix} c_\varphi c_\psi - s_\varphi c_\chi s_\psi & -c_\varphi s_\psi - s_\varphi c_\chi c_\psi & s_\varphi s_\chi \\ s_\varphi c_\psi + c_\varphi c_\chi s_\psi & -s_\varphi s_\psi + c_\varphi c_\chi c_\psi & -c_\varphi c_\chi \\ s_\chi s_\psi & s_\chi c_\psi & c_\chi \end{pmatrix}$$

Note: Euler angles are **NOT** vectors!

They lack the commutative property:

$v_1 + v_2 = v_2 + v_1$

In order to find the motion we need to integrate the accelerations and the velocities.

Newton: $\sum \mathbf{f}_c = m_c \ddot{\mathbf{x}}_c \rightarrow \ddot{\mathbf{x}}_c$

$\dot{\mathbf{x}}_c = \int \ddot{\mathbf{x}}_c \mathrm{d}t$, and
$\mathbf{x}_c = \int \dot{\mathbf{x}}_c \mathrm{d}t$

Euler: $\sum \mathbf{M}_c = \mathbf{I}_c \dot{\boldsymbol{\omega}}_c + \boldsymbol{\omega}_c \times (\mathbf{I}_c \boldsymbol{\omega}_c) \rightarrow \dot{\boldsymbol{\omega}}_c$

$\boldsymbol{\omega}_c = \int \dot{\boldsymbol{\omega}}_c \mathrm{d}t$, and
$(\varphi, \chi, \psi) =? \int \boldsymbol{\omega}_c \mathrm{d}t$: **NO!**

but
$(\varphi, \chi, \psi) = \int (\dot{\varphi}, \dot{\chi}, \dot{\psi}) \mathrm{d}t$

So what are angular velocities?

Angular velocities, for simplicity assume only rotation no translation ($\mathbf{t} = \mathbf{0}$):

$$\dot{\mathbf{p}} = \dot{\mathbf{R}}\mathbf{R}^T\mathbf{p}$$

The matrix $\dot{\mathbf{R}}\mathbf{R}^T$ is a-symmetric and therefore has 3 independent components, by definition the angular velocities $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$,

$$\dot{\mathbf{p}} = \tilde{\boldsymbol{\omega}}\mathbf{p} = \boldsymbol{\omega} \times \mathbf{p}$$

with the angular velocity matrix (=vector cross product)

$$\tilde{\boldsymbol{\omega}} = \left(\begin{array}{ccc} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{array} \right)$$

These angular velocities are vectors, contrary to the Euler angles.

The angular velocities in the body fixed frame are:

$$\dot{\mathbf{p}}' = \mathbf{R}^T\dot{\mathbf{R}}\mathbf{p}' = \tilde{\boldsymbol{\omega}}'\mathbf{p}'$$

or, since $\boldsymbol{\omega}$ is just a vector, simply transform:

$$\boldsymbol{\omega}' = \mathbf{R}^T\boldsymbol{\omega}$$

In order to integrate the motion we need an expression for the time derivatives of the Euler angles in terms of the angular velocities.

Start from

$$\dot{\mathbf{R}}\mathbf{R}^T = \tilde{\boldsymbol{\omega}} \rightarrow$$

$$\frac{\partial \mathbf{R}}{\partial \varphi}\mathbf{R}^T \dot{\varphi} + \frac{\partial \mathbf{R}}{\partial \chi}\mathbf{R}^T \dot{\chi} + \frac{\partial \mathbf{R}}{\partial \psi}\mathbf{R}^T \dot{\psi} = \tilde{\boldsymbol{\omega}}$$

or by means of successive rotation (visualize!):

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix} + \mathbf{R}_\varphi \begin{pmatrix} \dot{\chi} \\ 0 \\ 0 \end{pmatrix} + \mathbf{R}_\varphi \mathbf{R}_\chi \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} \rightarrow$$

combined

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 & c_\varphi & s_\varphi s_\chi \\ 0 & s_\varphi & -c_\varphi s_\chi \\ 1 & 0 & c_\chi \end{pmatrix} \begin{pmatrix} \dot{\varphi} \\ \dot{\chi} \\ \dot{\psi} \end{pmatrix}$$

But, we need the inverse expressions: given the Euler angles and the angular velocities what are the time derivates of the Euler angles:

$$\begin{pmatrix} \dot{\varphi} \\ \dot{\chi} \\ \dot{\psi} \end{pmatrix} = \frac{1}{s_\chi} \begin{pmatrix} -s_\varphi c_\chi & c_\varphi c_\chi & 1 \\ c_\varphi s_\chi & s_\varphi s_\chi & 0 \\ s_\varphi & -c_\varphi & 0 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Note the **singularity** at $\chi = 0 + k\pi$!

Moreover, any combination of angles f.i. Bryant angles, Cardan angles or whatever angles, will show a singular configuration somewhere along the line.

Finally, with the body fixed angular velocities

$$\mathbf{R}^T\dot{\mathbf{R}} = \tilde{\boldsymbol{\omega}}'$$

Resolve by means of successive rotation (visualize!):

$$\begin{pmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + \mathbf{R}_\psi^T \begin{pmatrix} \dot{\chi} \\ 0 \\ 0 \end{pmatrix} + \mathbf{R}_\psi^T \mathbf{R}_\chi^T \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix} \rightarrow$$

combine

$$\begin{pmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix} = \begin{pmatrix} s_\psi s_\chi & c_\psi & 0 \\ c_\psi s_\chi & -s_\psi & 0 \\ c_\chi & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\varphi} \\ \dot{\chi} \\ \dot{\psi} \end{pmatrix}$$

and the inverse expressions are .....

Alas, the Determinant equals s_χ, so again a singularity at at $\chi = 0 + k\pi$.

–The End–