

Lecture Notes
Multibody Dynamics B, wb1413

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April 16, 2009

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Chapter 8

Closed Loop Systems

Finding $\mathbf{x}_i = \mathbf{F}_i(\mathbf{q}_j)$ for closed loops is not easy. Lets look at \mathbf{f}_i a four-bar linkage. $\mathbf{x}_i = (x_1, y_1, \phi_1, x_2, y_2, \phi_2, x_3, y_3, \phi_3)$ and $\mathbf{q}_i = (\alpha)$.

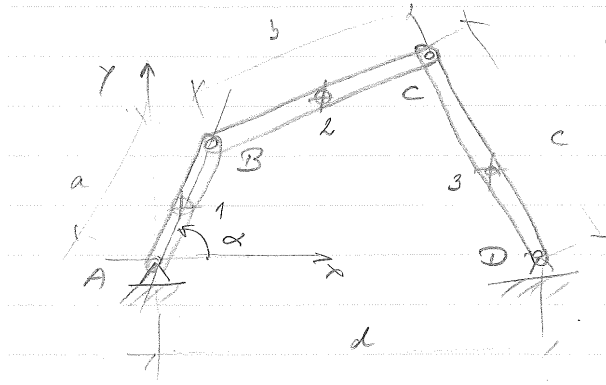


Figure 8.1: Four-bar linkage system

Why four-bar?

So we have to write down $\mathbf{x}_i = \mathbf{F}_i(\mathbf{q}_j)$.

Lets start,look at the figure above:

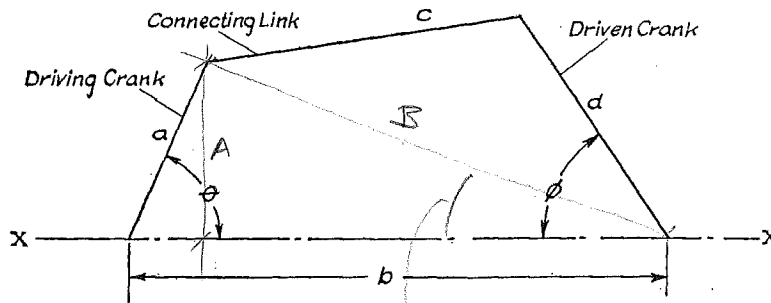
$$\begin{aligned}x_1 &= a/2 \cos(\alpha) \\y_1 &= a/2 \sin(\alpha) \\ \phi_1 &= \alpha \\x_2 &= a \cos(\alpha) + \dots\end{aligned}$$

I do not know! Look at the paper by Talbourdet from 1941 [1].

Part I—Analysis of Single 4-Bar Linkage

By Guy J. Talbourdet

Research Division
United Shoe Machinery Corporation



Calculate the values of the following factors from the geometry of the linkage:

$$K = a^2 + b^2 - c^2 + d^2$$

$$A = a \sin \theta$$

$$B = a^2 + b^2 - 2ab \cos \theta$$

$$D = K - 2ab \cos \theta$$

$$S = \sqrt{4d^2B - D^2}$$

θ = Angular displacement of driving crank.

ϕ = Angular displacement of driven crank.

$\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ = Angular velocity and acceleration respectively of the driving crank.

$\frac{d\phi}{dt}$ and $\frac{d^2\phi}{dt^2}$ = Angular velocity and acceleration respectively of the driven crank.

Then:

$$\phi = \tan^{-1} \frac{A}{b - a \cos \theta} + \cos^{-1} \frac{D}{2d\sqrt{B}}$$

And:

$$\frac{d\phi}{dt} = \frac{d\theta}{dt} \left[\frac{a}{B} (b \cos \theta - a) - \frac{Ab}{S} \left(2 - \frac{D}{B} \right) \right]$$

And:

$$\begin{aligned} \frac{d^2\phi}{dt^2} = & \frac{d^2\theta}{dt^2} \left[\frac{a}{B} (b \cos \theta - a) - \frac{Ab}{S} \left(2 - \frac{D}{B} \right) \right] \\ & + \left(\frac{d\theta}{dt} \right)^2 \left[\frac{2A^2b^2}{BS} \left(1 - \frac{D}{B} \right) + \left(2 - \frac{D}{B} \right) \left(\frac{2A^2b^2(2d^2 - D)}{S^3} - \frac{ab \cos \theta}{S} \right) \right. \\ & \left. - \frac{Ab}{B} \left(1 + \frac{2a(b \cos \theta - a)}{B} \right) \right] \end{aligned}$$

When the motion of the driving crank is uniform, $\frac{d\theta}{dt} = \omega$, and $\frac{d^2\theta}{dt^2} = 0$, in which case we have:

$$\frac{d\phi}{dt} = \omega \left[\frac{a}{b} (b \cos \theta - a) - \frac{Ab}{S} \left(2 - \frac{D}{B} \right) \right]$$

And:

$$\begin{aligned} \frac{d^2\phi}{dt^2} = & \omega^2 \left[\frac{2A^2b^2}{BS} \left(1 - \frac{D}{B} \right) + \left(2 - \frac{D}{B} \right) \left(\frac{2A^2b^2(2d^2 - D)}{S^3} - \frac{ab \cos \theta}{S} \right) \right. \\ & \left. - \frac{Ab}{B} \left(1 + \frac{2a(b \cos \theta - a)}{B} \right) \right] \end{aligned}$$

But I do know how to do a triple-pendulum. This is a double pendulum with an extra pendulum at the end and the I just add two constraints to get back to the original 1 dof system.

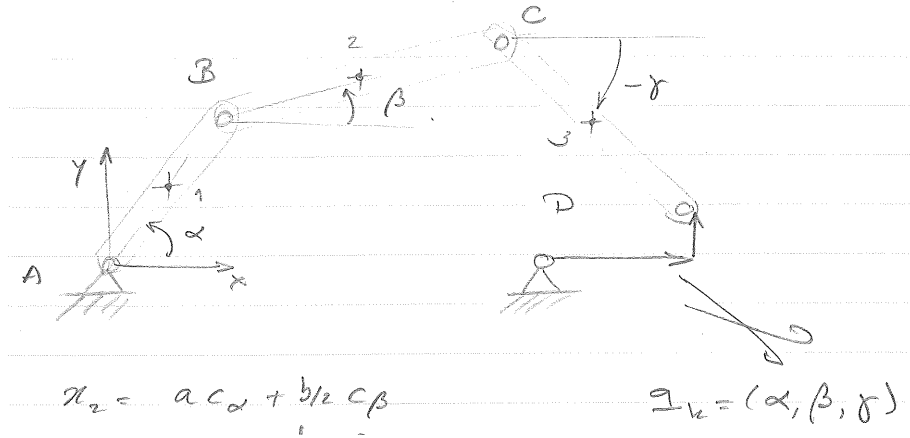


Figure 8.2: four-bar linkage generalized coordinate definition

So cut the loop at D and add two generalized coordinates β and γ . Next write down the positions and orientations of the rigid bodies in terms of the generalized coordinates, $\mathbf{x} = \mathbf{F}(\mathbf{q})$, where $\mathbf{q} = (\alpha, \beta, \gamma)$:

$$\begin{aligned}
 x_2 &= a \cos(\alpha) + \frac{b}{2} \cos(\beta) \\
 y_2 &= a \sin(\alpha) + \frac{b}{2} \sin(\beta) \\
 \phi_2 &= \beta \\
 x_3 &= a \cos(\alpha) + b \cos(\beta) + \frac{c}{2} \cos(\gamma) \\
 y_3 &= a \sin(\alpha) + b \sin(\beta) + \frac{c}{2} \sin(\gamma) \\
 \phi_3 &= \gamma
 \end{aligned}$$

Now add two constraints to close the loop again at D,

$$\begin{aligned}
 \epsilon_1 = \Delta x_D = 0 &\Rightarrow \epsilon_1 = a \cos(\alpha) + b \cos(\beta) c \cos(\gamma) - d = 0 \\
 \epsilon_2 = \Delta y_D = 0 &\Rightarrow \epsilon_2 = a \sin(\alpha) + b \sin(\beta) c \sin(\gamma) = 0
 \end{aligned}$$

And finally we can form the DAE for this problem as,

$$\begin{aligned}
 3\{ & \left[\begin{array}{cc} \mathbf{F}_{i,l} \mathbf{M}_{ij} & \mathbf{D}_{c,l} \\ \mathbf{D}_{c,k} & \mathbf{0} \end{array} \right] \left[\begin{array}{c} \ddot{\mathbf{q}}_k \\ \lambda_c \end{array} \right] = \left[\begin{array}{c} \mathbf{Q}_L + \mathbf{F}_{i,l} (\mathbf{f}_i - \mathbf{M}_{ij} \mathbf{g}_j) \\ -\mathbf{D}_{c,kl} \dot{\mathbf{q}}_k \dot{\mathbf{q}}_l \end{array} \right] \\
 2\{ &
 \end{aligned}$$

From this we solve for $\ddot{\mathbf{q}}_k$ and λ_c and then we integrate the state $\left[\begin{array}{c} \mathbf{q}_k \\ \dot{\mathbf{q}}_k \end{array} \right]$ like in:

$$\left[\begin{array}{c} \tilde{\mathbf{q}}_k \\ \tilde{\dot{\mathbf{q}}}_k \end{array} \right] = \left[\begin{array}{c} \mathbf{q}_k \\ \dot{\mathbf{q}}_k \end{array} \right] + \int_0^h \left[\begin{array}{c} \dot{\mathbf{q}}_k \\ \ddot{\mathbf{q}}_k \end{array} \right] dt$$

The approximate values at $t+h$ $\tilde{\mathbf{q}}_{n+1}$ and $\tilde{\dot{\mathbf{q}}}_{n+1}$ will in general NOT fulfill the constraints. Remember the results from the 2nd lecture things fly apart, see Figure 8.3. One can picture the constraints as a sort of surface in a higher dimensional space, where a state \mathbf{q} is represented by points in that space, see

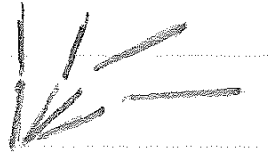


Figure 8.3: flying apart of the two bars of the pendulum

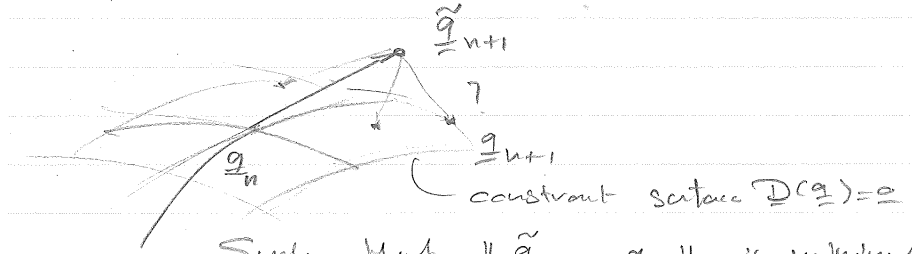


Figure 8.4: constraint surface

Figure 8.4. The constraint surface has the form $\mathbf{D}(\mathbf{q}) = \mathbf{0}$. Now a predicted solution $\tilde{\mathbf{q}}_{n+1}$ will in general not be on the constraint surface. We have to find a way to get back on the surface with minimal effort. Lets formulate this as a minimization problem such that the distance from the predicted solution $\tilde{\mathbf{q}}_{n+1}$ to the solution which is on the constraint surface is minimal: $\|\tilde{\mathbf{q}}_{n+1} - \mathbf{q}_{n+1}\|_2$ is minimal where all \mathbf{q}_{n+1} have to fulfill the constraints $\mathbf{D}(\mathbf{q}_{n+1}) = \mathbf{0}$

This is what we call a non-linear constrained least-square problem,

$$\begin{aligned} \|\tilde{\mathbf{q}}_{n+1} - \mathbf{q}_{n+1}\|_2 &= \min_{\mathbf{q}_{n+1}} \\ \mathbf{D}(\mathbf{q}_{n+1}) &= \mathbf{0} \end{aligned}$$

We solve this by a Gauss-Newton method: First linearize about $\tilde{\mathbf{q}}_{n+1}$.

$$\mathbf{q}_{n+1} = \tilde{\mathbf{q}}_{n+1} + \Delta \mathbf{q}_{n+1}$$

Which leads to:

$$\begin{aligned} \|\Delta \mathbf{q}_{n+1}\|_2 &= \min & \Sigma_i (\Delta q_i^2)_{n+1} &= \min \\ \mathbf{D}(\tilde{\mathbf{q}}_{n+1}) + \mathbf{D}_{,n}(\tilde{\mathbf{q}}_{n+1}) \Delta \mathbf{q}_{n+1} &= \mathbf{0} \end{aligned}$$

This constrained least square problem can be solved by introducing the so-called Lagrange multipliers $\boldsymbol{\mu}$ for the constraints leading to the linear system of equations,

$$\begin{bmatrix} \mathbf{I} & \mathbf{D}_{,q}^T \\ \mathbf{D}_{,q} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{q}_{n+1} \\ \boldsymbol{\mu} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -\mathbf{D}(\tilde{\mathbf{q}}_{n+1}) \end{bmatrix}$$

Or in a shorthand form, where we use $\boldsymbol{\Delta} = \Delta \mathbf{q}_{n+1}$, $\mathbf{D} = \mathbf{D}_{,q}$, and $\mathbf{e} = -\mathbf{D}(\tilde{\mathbf{q}}_{n+1})$,

$$\begin{bmatrix} \mathbf{I} & \mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Delta} \\ \boldsymbol{\mu} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{e} \end{bmatrix}$$

We have to solve for the vectors Δ and μ .

Start with $\Delta = -\mathbf{D}^T \mu$, and substitute this in the second equation, $\mathbf{D}\Delta = \mathbf{e}$, as,

$$-\mathbf{D}\mathbf{D}^T \mu = \mathbf{e}$$

Note the dimension of the matrix \mathbf{D} ($m \times n$) where $m < n$ and the product $\mathbf{D}\mathbf{D}^T$ ($m \times m$) which is now square in the smallest dimension m . If this matrix has full rank, which it usually will have, then we can solve for μ and \mathbf{e} ,

$$\begin{aligned} \mu &= -(\mathbf{D}\mathbf{D}^T)^{-1} \mathbf{e} \\ \Delta &= \mathbf{D}^T (\mathbf{D}\mathbf{D}^T)^{-1} \mathbf{e} \end{aligned}$$

For an undetermined linear system of equations with full rank matrix \mathbf{D} , the matrix,

$$\mathbf{D}^+ = \mathbf{D}^T (\mathbf{D}\mathbf{D}^T)^{-1}$$

is called the Moore-Penrose pseudo inverse and gives us the least square solution of the problem.

Example:

$$\begin{array}{rcl} x_2 - x_1 & = & 0 \\ x_2 - x_3 & = & 0 \end{array} \quad \text{with values} \quad \begin{array}{l} x_1 = 0.9 \\ x_2 = 1 \\ x_3 = 1 \end{array}$$

The equations are linear so the Jacobian, \mathbf{D}, \mathbf{x} is simply the matrix,

$$\begin{aligned} \mathbf{D} &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} & \tilde{\mathbf{x}} &= \begin{bmatrix} 0.9 \\ 1 \\ 1 \end{bmatrix} \\ \mathbf{D}\tilde{\mathbf{x}} &= \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} & & \text{"errors"} \end{aligned}$$

$$\begin{aligned} \mathbf{x} = \tilde{\mathbf{x}} + \Delta \mathbf{x} &\rightarrow \mathbf{D}\tilde{\mathbf{x}} + \mathbf{D}\Delta \mathbf{x} = 0 \\ \mathbf{D}\Delta \mathbf{x} = -\mathbf{D}\tilde{\mathbf{x}} &\rightarrow \Delta \mathbf{x} = \mathbf{D}^+ (-\mathbf{D}\tilde{\mathbf{x}}) \end{aligned}$$

$$\mathbf{D}^+ = \mathbf{D}^T (\mathbf{D}\mathbf{D}^T)^{-1}$$

$$\mathbf{D}\mathbf{D}^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$(\mathbf{D}\mathbf{D}^T)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{D}^+ = \mathbf{D}^T (\mathbf{D}\mathbf{D}^T)^{-1} = \frac{1}{3} \begin{bmatrix} -2 & -1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\Delta \mathbf{x} = \mathbf{D}^+ (-\mathbf{D}\tilde{\mathbf{x}}) = \frac{1}{3} \begin{bmatrix} -2 & -1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.06 \\ -0.03 \\ -0.03 \end{bmatrix}$$

$$|\Delta \mathbf{x}| = 0.0816$$

$$\mathbf{x} = \begin{bmatrix} 0.9 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.06 \\ -0.03 \\ -0.03 \end{bmatrix} = \begin{bmatrix} 0.96 \\ 0.97 \\ 0.97 \end{bmatrix}$$

Note that this solution (which is on the constraint surface) is really at the shortest distance from the approximate solution (0.9, 1, 1). You can easily come up with other solutions which are on the surface, like (1, 1, 1), but they are always farther away (check this).

The Gauss-Newton iteration scheme is now,

```

set iterat = 0
set tol = 1e-12
set x_n+1 x_n
set maxiterat = 10
evaluate eps=D(x_n+1)
repeat
  dx_n+1 = -D,x^T(D,x D,x^T) eps
  x_n+1 = x_n+1 + dx_n+1
  eps = D(x_n+1)
  iterat = iterat+1
until max(abs(eps))<tol or iterat>maxiterat

```

Next we determine the speeds which fulfill the constraints, these are linear equations so we have a linear least square problem which we can solve in one step:

```

epsdot=D,x xdot_n+1
dxdot_n+1 = -D,x^T(D,x D,x^T) epsdot
xdot_n+1 = xdot_n+1 + dxdot_n+1

```

Now take a look at the Hrones & Nelson [2] four-bar linkage atlas. A 700 page folio book from 1951, which shows 500.000 solutions of the coupler curve for a general four-bar linkage. And how are these constructed? with a mechanism!

Bibliography

- [1] G. J. Talbourdet. Mathematical Solutions of Four-Bar Linkages; part I - Analysis of Single 4-Bar Linkage. *Machine Design*, May 1941.
- [2] J. A. Hrones and G. L. Nelson. *Analysis of the four bar linkage*. MIT press and John Wiley & Son, New York, 1951.